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VOLUME I

A RATIONAL METHOD FOR PREDICTING ALIGHTING GEAR DYNAMIC LOADS

VOLUME I
GENERAL METHODS

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Volume I

FOREWORD

The research work in this report was performed by Chance Vought Corporation, Dallas, Texas, for the Vehicle Dynamics Division, AF Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, under Contract AF33(616)-7824. This research is part of a continuing effort to provide a more rational and reliable theoretical method for establishing design criteria in the specific area of shock and impact for flight vehicles and is part of the Research and Technology Division, Air Force Systems Command's exploratory development program. The Department of Defense Program Element number is 6.24.05.33.4, "Aircraft Flight Dynamics". This work was performed under Project No. 1367, "Structural Design Criteria" and Task No. 136706, "Prediction and Measurement of Structural Dynamic Loads Including Fatigue Aspects". Mr. W. P. Dunn of the AF Flight Dynamics Laboratory was the Project Engineer. The research was conducted from 2 February 1961 to 30 June 1962 by the Structures Section of the Aero Division of Chance Vought Corporation.

The report is Volume I of a two volume report which presents the formulation of the equations of motion and defines in detail many applied forces of present and future vehicles necessary for solution of the equations of motion formulation, Volume II - Examples, presents illustrative examples with accompanying numerical solutions so that the procedure for manipulating the equations of motion formulated in Volume I can be used as a guide when the method is utilized.

Volume I contains Sections 1 through 3, Appendixes A through D, and the Bibliography. Volume II contains Sections 4 and 5.

ABSTRACT

A rational method for predicting alighting gear loads during landing impact is discussed. The equations describing the motions of a vehicle during landing impact are developed for an arbitrary vehicle configuration. The method is of sufficient generality and accuracy to allow the formulation of alighting gear dynamics problems in flight vehicles including V/STOL aircraft, high gross weight logistic vehicles, recoverable booster vehicles, advanced tactical and defense vehicles operating out of remote areas, and lunar vehicles. It allows for the effects of varying coefficients of friction and damping, combinations of initial conditions of pitch, roll, yaw angles and rates, vertical, longitudinal and lateral motion, slippage of the gear relative to the alighting surface, flexible alighting gear and vehicle structure, simultaneously applied triaxial ground loads, and various types and number of alighting elements. The formulation is intended for the cantilevered type of gear, although the articulated type may be handled through some extensions of the formulation. A survey of the various types of forces which occur during landing impact is made, and the manner in which these forces enter the equations of motion is described.

The general equations may be reduced for a particular problem by imposing the vehicle configuration and any simplifying assumptions directly on the equations. Several illustrative examples with accompanying numerical solutions are provided in Volume II, "Examples".

The report may be used as a guide in the formulation of a landing impact problem.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE DIRECTOR

Howard A. Magrath

HOWARD A. MAGRATH
Chief, Vehicle Dynamics Division
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LIST OF SYMBOLS

$[A]$	The mass matrix for the main component of the vehicle, in which inertial effects of the appended components are included.
A	The cross-sectional area normal to the direction of motion of a body impacting in soil.
$\bar{A}$	The tire contacting or footprint area.
a	Paragraph 3.2.2 - A point on the thrust line; one choice is the point of intersection of the engine centerline with the rear engine bulkhead plane.
a	Paragraph 3.3.3.2 - The value of the hydraulic oil bulk modulus at zero pressure.
a	Paragraph 3.5 - One of a pair of simple beams.
a	Paragraph 4.2.1 - One of a pair of contacting elements.
A_A	The pneumatic area in a pneumatic spring.
AG	The shear stiffness in a beam.
A_H	The hydraulic area in a hydraulic damper.
A_{HS}	The hydraulic area during the return stroke.
$\dot{A}_N$	The orifice area.
A_P	The piston area.
A_R	The rod cross-sectional area in a liquid spring.
$[A_{x'x'}], [A_{y'y'}], [A_{z'z'}]$ $[A_{x'y'}], [A_{x'z'}], [A_{y'x'}]$ $[A_{y'z'}], [A_{z'x'}], [A_{z'y'}]$	The mass matrices describing the inertial properties of a vehicle component which arise from terms in the kinetic energy involving products of the panel point velocities or displacements. The subscripts refer to the directions of the corresponding velocities or displacements.
$[A_{x'}]$ $[A_{y'}]$ $[A_{z'}]$	The mass matrices describing the inertial properties of a vehicle component which arise from terms in the kinetic energy involving the product of a variable governing the vehicle motion with panel point displacements or velocities parallel to the x', y' and z' axes respectively.

$[x'A_y], [x'A_z]$, $[y'A_x], [y'A_z]$, $[z'A_x], [z'A_y]$	The mass matrices describing the inertial properties of a vehicle component which arise from terms in the kinetic energy involving the product of the components of the position vector $\mathbb{L}$ with the panel point velocities or displacements. The symbol to the left refers to the component of $\mathbb{L}$, and the subscript to the right refers to the component of panel point velocity or displacement.
a_{11}, a_{12}	Modal masses for the first two wing modes.
a_{21}, a_{22}	Modal mass moments for the first two wing modes.
b	Paragraph 3.2.2 - A point on the thrust line; one choice is the point of intersection of the engine centerline with the forward engine bulkhead plane.
b	Paragraph 3.3.3.2 - The derivative of the hydraulic oil bulk modulus with respect to pressure.
b	Paragraph 3.5 - One of a pair of simple beams.
b	Paragraph 4.2.1 - One of a pair of contacting elements.
B	The bulk modulus of soil.
B_1, B_2	The bulk modulus of oil in the sections of a liquid spring.
C_a, C_b	The damping rates of the simple spring-dampers supporting the contacting elements a, b.
C_c	The coefficient of contraction.
C_D	The orifice coefficient; the product of C_c with C_v
C_i	One of the aerodynamic coefficients.
C_l, C_m, C_n	Non-dimensional aerodynamic moment coefficients.
C_p	The local pressure coefficient.
C_v	The coefficient of velocity; the ratio of the actual hydraulic oil velocity to that given by Bernoulli's relation.
c_x, c_y, c_z	Non-dimensional aerodynamic coefficients.
d	Paragraph 3.2 - The ground penetration depth of a contacting element.
d	Paragraph 3.4.3 - The damping of a modal coordinate.
$\lceil d \rfloor$	The modal damping matrix.

$[D]$	The panel point damping matrix.
D	Paragraph 3.2.5.3 - The spike diameter.
D	Paragraph 3.2.5.4 - The ski or skid planing length.
D_{BN}, D_{BM}	The piston bottoming force damping rates for the nose and main gears.
d_c	The critical damping of a modal coordinate.
D_G	The force on a contacting element from the ground, parallel to the X axis.
d_G	The piston length plus the tire radius.
D_{GN}, D_{GM}	The ground forces parallel to the X axis acting on the nose and main gear tires.
d_N, d_M	The damping rates for nose and main gear elastic motions with a single degree of freedom.
$\{d_P\}$	The distances from the upper piston bearing to the piston panel points.
$[E]$	The matrix of influence coefficients; the inverse of the stiffness matrix.
EI	The bending stiffness.
F_A	The pneumatic force.
F_B	The piston bottoming force which retains the piston in the cylinder.
F_{BL}, F_{BU}	The lower and upper bearing forces.
$F_{BLx'}, F_{BLy'}$	Components of the lower bearing force.
$F_{BUx'}, F_{BUy'}$	Components of the upper bearing force.
F_C	Contribution to the stroking force from a crushable material.
F_{Da}, F_{Db}	The damping forces in the support members for contacting elements a, b.
F_F	The bearing friction force.
F_H	The hydraulic force.
F_i	A contraint relationship.
F_{NL}, F_{NU}	The lower and upper seal normal forces in a liquid spring system.

f_o	The constant force characterizing the crushable material shock absorbing mechanism.
$[F_{px'}],[F_{py'}]$	The piston panel point loads, including applied forces and inertial reactions.
F_s	The spring force.
F_{sa}, F_{sb}	The spring forces in the support members for contacting elements a, b.
f_{s1}, f_{s2}	The functional form of the force-deflection characteristics of a mechanical spring.
F_{xDM}, F_{xDN}	The damping forces on the main and nose gear fore and aft elastic motions.
$f(\Delta_z)$	The metering function which multiplies the square of the stroking velocity to yield the hydraulic force.
g	The acceleration due to gravity.
h	The maximum value of stroke.
$\mathbb{I}$	The unit vector along the X axis of the ground coordinate system.
i	The unit vector along the x axis of the body coordinate system.
i'	The unit vector along the x' axis of a component coordinate system.
i''	The unit vector in the articulated gear coordinate system, parallel to the body x axis.
I_A	The moment of inertia of a component about the line around which it rotates as a rigid body.
I_{PP}	The moment of inertia of the piston, wheel, tire, and brakes about the gear centerline.
I_{xx}, I_{yy}, I_{zz}	The moments of inertia about the body axes of the vehicle in its undeflected position.
$\mathbb{J}$	The unit vector along the Y axis of the ground coordinate system.
j	The unit vector along the y axis of the body coordinate system.
j'	The unit vector along the y' axis of a component coordinate system.

$\mathbb{J}''$	The unit vector along the y'' axis of the articulated gear coordinate system, pointing outboard along the tension strut centerline.
$\mathbb{K}$	The unit vector along the Z axis of the ground coordinate system.
$\mathbb{K}$	The unit vector along the z axis of the body coordinate system.
$\mathbb{K}'$	The unit vector along the z' axis of a component coordinate system.
$\mathbb{K}''$	The unit vector along the z'' axis of the articulated gear coordinate system.
K	Paragraph 3.2.5.2 - The tire cornering coefficient.
k_a, k_b	The spring rates in the support members of contacting elements a, b.
K_B	The piston bottoming spring rate.
K_β	The torsional spring rate of the cantilevered gear about the gear centerline.
K_{BL}, K_{BU}	The lower and upper piston bottoming spring rates.
K_{BM}, K_{BN}	The lower piston bottoming spring rates of the main and nose gears.
$[K_N]$	The stiffness matrix defining the normal modes of vibration.
K_R	The spring rate of the relief valve spring.
K_{SN}	Metering function for the return stroke.
K_{TM}, K_{TN}	The tire spring rates of the main and nose gear tires.
K_{xM}, K_{xN}	The main and nose gear fore and aft spring rates at the axles.
$K_{xM1}, K_{xM2}, K_{xM3}$	The main gear fore and aft axle spring rate polynomial coefficients.
$K_{x'1}, K_{x'2}, K_{x'3}$	The cantilevered gear fore and aft axle spring rate polynomial coefficients.
$K_{y'1}, K_{y'2}, K_{y'3}$	The cantilevered gear lateral axle spring rate polynomial coefficients.
$K_{x'x'}, K_{y'y'}$	The cantilevered gear axle spring rates which are considered as functions of stroke.

$[K_{x'x'}], [K_{y'y'}], [K_{z'z'}]$	The stiffness matrices defining the potential energy due to elastic deformations of a component along the x', y' and z' axes respectively.
$[K_{x'y'}], [K_{x'z'}]$ $[K_{y'x'}], [K_{y'z'}]$ $[K_{z'x'}], [K_{z'y'}]$	The stiffness matrices arising in the type of structure in which an applied force in one direction causes a deflection at right angles to that direction. These matrices are not retained in the equations of motion except in the appendix, as they do not occur for normal structures.
$\mathbb{L}$	The position vector from the origin of the body coordinate system to the undeflected position of an elemental mass.
L	The Lagrangian; kinetic energy minus potential energy.
ℓ	The separation of the spike point from the spike center of mass.
$\{l\}$	The distances to a set of panel points from the y' axis about which a component may rotate as a rigid body.
$\mathbb{L}_A$	The position vector of the axle in the undeflected vehicle.
$\bar{\ell}$	The distance to the center of mass of a component which may rotate as a rigid body about a line, measured from the line.
l_A	The axle offset length.
l_B	The bearing separation at zero stroke.
l_b	The distance from the cantilevered gear centerline to the support strut connection point.
l_j	The length of the j-th bay in a simple beam.
L_M, L_N	The separations of the upper bearing and the axle of the main and nose gears.
l_p	The separation of the upper bearing and the axle of the cantilevered gear.
$\mathbb{L}_T$	A vector collinear with the thrust vector.
l_{UM}, l_{UN}	Initial bearing separation in the main and nose gears.
L_1	The distance from the upper bearing to the contact pad.
L_2	The distance from the lower bearing to the contact pad before stroking occurs.
$[L_1]$	The transformation from panel point loads and reaction forces to panel point shears and moments.

$[\bar{L}_1][\bar{L}_2]$	Matrices formed by partitioning $[L_1]$
$[L_3]$	The transformation from panel point loads to support reactions.
$[L_4]$	The transformation from panel point loads to panel point shears and moments.
M	The vehicle mass.
M_ℓ	The mass of a component of the vehicle which may move as a rigid body relative to the body axes (Δ - motion).
M_i	The mass of the i-th component.
M_j	The bending moment in the j-th bay.
M_M, M_N	The masses of the main and nose gears.
M_p	The mass of the wheel, tire, brakes, and piston in the cantilevered gear.
$\bar{M}_p$	The mass of the wheel, tire, and brakes in the cantilevered gear.
M_u	The unsprung mass.
M_1, M_2	The masses supported by weightless flexible beams.
n	The exponent which indicates the exact polytropic nature of the pneumatic compression.
N_{AM}, N_{AN}	Spinup moments on the main and nose gear tires.
$N_{p\beta}$	The sum of the applied moments and restraining (scissors) spring moment on the torsional motion of the cantilevered gear unsprung mass.
N_{SU}	The spinup moment.
N_{WG}	The moment on the cantilevered gear axle due to the force $Q_{y'G}$, which acts normal to the wheel plane.
N_x, N_y, N_z	The components of the total applied moment on the vehicle.
N_{xG}, N_{yG}, N_{zG}	The components of the applied moment on the vehicle due to the ground force on a contacting element.
N_{ypG}	The moment on the spike about its center of mass due to ground forces.
N_ψ, N_θ, N_ϕ	Components of the total applied moment on the vehicle. These are non-orthogonal components, as they act about the z axis, the line of modes, and the x axis respectively.

$N(\eta, \dot{\eta})$	The bogie, ski, or skid restraining moments.
$\mathbb{P}$	The total displacement of an elemental mass in the vehicle.
$\{P\}$	The symbolic form for the panel point displacements.
$\mathbb{P}_A$	The axle total displacement.
$\bar{P}_A$	Pressure in the airchamber of the pneumatic spring.
$\bar{P}$	The undeflected tire pressure.
P_c	The critical pressure in a gas chamber with relief valve.
$\bar{P}_c$	The tire wall equivalent pressure.
P_{cT}	The average tire contacting pressure.
$\overline{\Delta P}$	Tire pressure rise on deflection.
$\mathbb{P}_i$	The total displacement of an elemental mass in the i-th component of the vehicle.
$\mathbb{P}_i^e$	The displacement due to elasticity of an elemental mass in the i-th component of the vehicle.
$\mathbb{P}_{ij}$	The displacement of an elemental mass in the i-th component due to displacement of the j-th component, to which the i-th component is affixed.
P_L	The displacement of the lower bearing in the cantilevered gear, at right angles to the gear centerline.
$P_{Lx'}, P_{Ly'}$	The components of P_L
P_U	The displacement of the upper bearing in the cantilevered gear, at right angles to the gear centerline.
$P_{Ux'}, P_{Uy'}$	The components of P_U.
$\mathbb{P}_W$	The vector displacement of the instantaneous center of mass of the vehicle from the origin of the body coordinate system.
$P_{x'}, P_{y'}, P_{z'}$	The components of $\mathbb{P}_i$.
$\{P_{x'}\}, \{P_{y'}\}, \{P_{z'}\}$	The panel point displacements parallel to the component coordinate axes.
$\{P_{x'}^e\}, \{P_{y'}^e\}, \{P_{z'}^e\}$	The panel point displacements due to elasticity.
$\{P_{x'}\}_{ij}$, $\{P_{y'}\}_{ij}$, $\{P_{z'}\}_{ij}$	The panel point displacements in the i-th component due to displacement of the j-th component, to which the i-th component is affixed.

$P_{x'pij}, P_{y'pij}$	Components of the displacement of panel point i on the piston due to displacement of the cylinder.
P_{xM}, P_{xN}	The axle fore and aft displacements of the main and nose gears.
P_1, P_2	Pressures in the two regions of a liquid spring.
$\mathcal{Q}$	The total applied force on the vehicle.
$\{q\}$	A column vector of modal coordinates.
$\{\bar{q}\}$	A column vector of modal coordinates with the higher modes deleted.
$\mathcal{Q}_A$	The aerodynamic force on the vehicle.
Q_i	The generalized force associated with the generalized coordinate q_i
q_i	A generalized coordinate.
$\mathcal{Q}_P$	The parachute force on the vehicle.
$\mathcal{Q}_T, Q_T$	The thrust vector and its magnitude.
$\mathcal{Q}_W$	The gravitational force on the vehicle.
Q_X, Q_Y, Q_Z	The components of the total vehicle force along the axes of the ground coordinate system.
Q_x, Q_y, Q_z	The components of the total vehicle force along the axes of the body coordinate system.
$\{Q_{x'}\}, \{Q_{y'}\}, \{Q_{z'}\}$	The panel point applied force components.
$\{\hat{Q}_{x'}\}, \{\hat{Q}_{y'}\}, \{\hat{Q}_{z'}\}$	The panel point load components; the difference between the applied forces and the inertial reactions.
Q_{xG}, Q_{yG}, Q_{zG}	The components of the ground force on the vehicle.
Q_{xGa}, Q_{yGa} Q_{xGb}, Q_{yGb}	Two components of the ground force on the contacting elements a, b,
Q_{xGM}, Q_{zGM} Q_{xGN}, Q_{zGN}	Two components of the ground force on the main and nose gear tires.
$Q_{x'p}, Q_{y'p}, Q_{z'p}$	Components of the total applied force on the piston.
$Q_{z'GP}$	The ground force acting along the piston z'-axis.
Q_{zA}	The aerodynamic lift.

$Q_{z'S}$	The total stroking force; the sum of the hydraulic, pneumatic, bearing friction, and bottoming forces.
Q_{zSM}, Q_{zSN}	The total stroking forces in the main and nose gears.
Q_{xw}, Q_{yw}, Q_{zw}	The components of the gravitational force.
$\mathbb{R}$	The position vector of the origin of the body coordinate system.
$\mathbb{r}$	The position vector of an elemental mass in the vehicle relative to the ground coordinate system.
$[R]$	The transformation from the Eulerian angle time derivatives to the components of the angular velocity expressed in the body coordinate system.
$\bar{r}$	The tire undeflected radius.
R_1, R_2	On the cylinder; components of the support reaction along and at right angles to the gear centerline. On the piston, the bearing forces.
$(C_D S)$	The effective drag area of the parachute. The symbols may be separately defined as the drag coefficient and area, but no standard definition holds (See Ref. 5). The effective area is generally measured for each parachute.
s	The stroke of the piston; the displacement of the piston relative to the cylinder, positive as the piston enters the cylinder.
S_G	The component of the ground force on a contacting element parallel to the ground Y axis.
T	The kinetic energy of the vehicle.
t	Time
$[T_{BC}]$	The matrix relating the displacements at the bearing points to the displacements of the panel points on the cylinder. The elements of this matrix are determined by the stroke, the position of the panel and the interpolation scheme used to relate the cylinder elastic displacement to the panel point displacements.
$[T_{ij}]$	The matrix yielding the panel point displacements of the i-th component as a rigid body due to displacement of the j-th component, to which the i-th component is affixed.
$[T_{jF}]$	The matrix yielding the panel point displacements of the j-th component as a rigid body due to displacement of the fuselage, to which the j-th component is affixed.

$[T_{MV}]$	The general geometrical transformation between shears and moments, and the panel point applied forces and reactions.
$[T_{PC}]$	The matrix yielding the panel point displacements of the piston as a rigid body due to displacements of the cylinder (or the bearing points).
t_t	The spinup time.
$[T_{TF}]$	The matrix yielding the panel point displacements of the tail as a rigid body due to fuselage panel point displacements.
$[T_{VH}]$	The matrix yielding the panel point displacements of the vertical beam as a rigid body due to panel point displacements of the horizontal beam.
$[T_{WF}]$	The matrix yielding the panel point displacements of the wing as a rigid body due to panel point displacements of the fuselage.
U	The potential energy due to elastic deformations.
V	Volume.
v	The velocity of propagation of a compression wave.
$\mathcal{V}$	The velocity of the vehicle relative to the atmosphere.
ν	The velocity of the metered hydraulic oil obtained from Bernoulli's relation.
$[\nu]$	Defined by Eq. 2.7-12.
$\vec{V}_B$	The velocity of a point B in the vehicle, relative to the ground.
V_G	The component of the ground force on a contacting element parallel to the z axis.
V_{GM}, V_{GN}	The ground force V_G on the main and nose gear tires.
V_j	The panel point shear load at the j-th bay.
V_{LA}, V_{NA}	The components of the ski or skid axle velocity at right angles to the element.
V_M	The volume of oil metered through the orifice.
V_0	The initial value of the volume.
$\mathcal{V}_P$	The velocity of the parachute attachment point relative to the atmosphere.

v_R	The speed of the tire footprint relative to the ground.
v_x, v_y, v_z	The components in the body coordinate system of the vehicle velocity.
V_{xB}, V_{yB}, V_{zB}	The components of the velocity of a point B on the body relative to the ground.
v_{xw}, v_{yw}, v_{zw}	The components in the body coordinate system of the wind velocity.
V_{10}, V_{20}	The initial volumes of the two regions of the liquid spring.
$\frac{\partial V_{1c}}{\partial P_1}, \frac{\partial V_{1c}}{\partial \Delta}, \frac{\partial V_{1s}}{\partial P_1}$	The rates at which the volume V_1 of the liquid spring cylinder changes with pressure and stroke due to cylinder expansion and seal compression.
w	The tire width.
W_D, W_L, W_N	The components of the ground force on a ski or skid in the coordinate system of the element.
W_S	The component of the ground force on a tire, in the ground plane and normal to the line of intersection of the wheel plane and the ground plane.
W_{SU}	The spinup force; the component of the ground force on a tire, in the ground plane and parallel to the line of intersection of the wheel plane and ground plane.
X	A component of the position vector $\mathbb{R}$ in the ground coordinate system.
x	A component of the position vector $\mathbb{L}$ in the body coordinate system.
x'	A component of the position vector $\mathbb{L}$ in a component coordinate system.
$\bar{x}'$	A component of the position vector $\mathbb{L}$ to the center of mass of a vehicle component, expressed in the component coordinate system.
$\{x'\}$	The panel point positions along the x'-axis.
$\dot{X}$	The component of the vehicle velocity along the X -axis.
$\bar{x}_a$	The component of the position vector to the contacting element 'a' along the x-axis.
$\bar{x}_{AM}, \bar{x}_{AN}$	The components of the position vector $\mathbb{L}$ to the nose and main gear axles along the x-axis.

$\dot{X}_B$	The component of the velocity of a point B in the vehicle, relative to the ground and along the X-axis.
$\dot{X}_e$	The component of the velocity of the trailing end of a ski or skid, relative to the ground and along the X-axis.
X_o	The initial value of the component of $\mathbb{R}$ along the X-axis.
$\dot{X}_P$	The component of the pad velocity relative to the ground along the X-axis.
$\{x\}_w$	The panel point coordinates of the wing panel points.
x_{WR}	The coordinate of the wing root.
Y	A component of the position vector $\mathbb{R}$ in the ground coordinate system.
y	A component of the position vector $\mathbb{L}$ in the body coordinate system.
y'	A component of the position vector $\mathbb{L}$ in a component coordinate system.
$\bar{y}'$	A component of the position vector $\mathbb{L}$ to the center of mass of a vehicle component, expressed in the component coordinate system.
$\{y'\}$	The panel point positions along the y'-axis.
$\dot{Y}$	The component of the vehicle velocity along the Y-axis.
$\dot{Y}_B$	The component of the velocity of a point B in the vehicle, relative to the ground and along the Y-axis.
$\dot{Y}_e$	The component of the velocity of the trailing end of a ski or skid, relative to the ground and along the Y-axis.
Y_o	The initial value of the component of $\mathbb{R}$ along the Y-axis.
$\dot{Y}_P$	The component of the pad velocity relative to the ground along the Y-axis.
Z	The distance from the ground to the origin of the body coordinate system, normal to the ground.
z	A component of the position vector $\mathbb{L}$ in the body coordinate system.
z'	A component of the position vector $\mathbb{L}$ in a component coordinate system.
$\bar{z}'$	A component of the position vector $\mathbb{L}$ to the center of mass of a vehicle component, expressed in the component coordinate system.

$\{z'\}$	The panel point positions along the z'-axis.
$\dot{z}$	The component of the vehicle velocity normal to the ground plane.
z_A	The axle height above the ground.
$\bar{z}_{AM}, \bar{z}_{AN}$	The main and nose gear axle coordinates along the z-axis.
z_B	The height of point B above the ground.
$\dot{z}_e$	The velocity of the trailing end of a ski or skid, relative to the ground and along the z-axis.
$\dot{z}_p$	The pad velocity relative to the ground along the z-axis.
z_o	The initial value of the component of R along the z-axis.
z_o	The distance along the spike measured from the point.
α	Paragraph 3.2.4 - The vehicle angle of attack.
α	Paragraph 4.3.5.1 - The angle at which the cantilevered gear shock strut is canted forward from the z-axis.
α	Paragraph 4.2.5.2 - The angle of rotation of the articulated gear.
α_A	The angle between the wheel axle and the ground plane.
β	Paragraph 3.2.4 - The vehicle sideslip angle.
β	Paragraph 3.2.5.2 - The angle through which the wheel axle is rotated about the gear centerline due to torsional elasticity.
β	Paragraph 3.2.5.4 - The angle through which a ski or skid rotates about the gear centerline due to torsional elasticity.
β	Paragraph 3.2.5.3 - The spike apex angle.
$[\Gamma']$	The transformation from the ground coordinate system to the body coordinate system.
$[\gamma]$	The transformation from a component coordinate system to the body coordinate system.
γ_A	The angle between the articulated gear tension strut centerline and the wheel axle.
γ_D	The angle between the support strut and the gear centerline.

$\gamma_{xT}, \gamma_{yT}, \gamma_{zT}$	The cosines of the angles between the thrust vector and the axes of the body coordinate system.
δ	Paragraph 3.2.4 - A control surface deflection.
δ	The tire deflection.
δ_A	A roll control surface deflection, such as aileron or spoiler.
δ_b	The tire deflection at bottoming.
δ_E	A pitch control surface deflection.
δ_F	A flap deflection.
Δ'_i	The displacement of an elemental mass from its undeflected position defining component rigid body displacement with respect to the body axes, expressed in the component coordinate system in which the motion is most easily described.
δ_R	A rudder deflection.
δ_{TM}, δ_{TN}	The main and nose gear tire deflections.
$\{\Delta_{x'}\}, \{\Delta_{y'}\}, \{\Delta_{z'}\}$	The panel point displacement components due to Δ-motion of the component.
$\Delta_{z'}$	The rigid body displacement along the stroking axis (Δ-motion along a line).
Δ_{za}, Δ_{zb}	The displacements along the stroking axes of the contacting elements a, b.
Δ_{zM}, Δ_{zN}	The displacements along the stroking axes of the main and nose gear pistons.
ζ_ω	The fraction of critical damping of the mode of frequency ω.
η	The angle defining component rigid body motion about a line.
$\dot{\eta}$	The angular velocity of a component rotating as a rigid body about a line.
$\dot{\eta}_{ASU}$	Wheel spinup angular velocity.
$\dot{\eta}_{MSU}, \dot{\eta}_{NSU}$	The main and nose gear wheel angular velocities at spin-up.
θ	One of the Euler angles defined in Paragraph 2.7.

θ_0	The initial value of θ.
λ	The angle between the component of axle velocity parallel to the ground and the line of intersection of the wheel plane and the ground plane.
λ_F	An eigenvalue of the fuselage homogeneous panel point equations.
λ_i	An eigenvalue corresponding to the i-th mode of elastic vibration.
μ	The coefficient of friction between a contacting element and the ground.
μ_b	A bearing coefficient of friction.
μ_{BL}, μ_{BU}	The lower and upper bearing coefficients of friction.
μ_L, μ_U	The lower and upper liquid spring seal coefficients of friction.
ξ	A dimensionless local bay coordinate used to non-dimensionalize some forms in the interpolation schemes.
ρ	The vehicle mass density; or, the atmospheric density.
ρ_H	The hydraulic fluid density.
$\{\sigma\}$	A column matrix of undetermined multipliers.
σ_j	The undetermined multiplier associated with the constraint relation F_j.
ϕ	One of the Euler angles defined in Paragraph 2.7.
$\{\phi\}$	A column matrix defining a mode shape; the elements are proportional to the actual panel point displacements in the mode.
$[\phi]$	A square matrix of modal columns $\{\phi\}$
$[\tilde{\phi}]$	A rectangular matrix of modal columns in which the higher modes are deleted.
ϕ_0	The initial value of ϕ.
ψ	One of the Euler angles defined in Paragraph 2.7.
ψ_0	The initial value of ψ.
ω	The frequency of vibration of a mode.
Ω	The vehicle angular velocity.

$[\Omega]$	Defined by Eq. 2.7-11.
$\Omega_x, \Omega_y, \Omega_z$	The components of Ω in the body coordinate system.
$\Omega_{x'}, \Omega_{y'}, \Omega_{z'}$	The components of Ω in a component coordinate system.

Subscripts

A Axle; axis

a A point in the vehicle; a contacting element

B A point in the vehicle

b A contacting element

c Cylinder

e Ski or ski trailing end

F Fuselage

f Final value

G Ground

H Horizontal beam

i One of the vehicle components, panel points, modal coordinates, etc. This subscript must be determined in context.

j One of the vehicle components, panel points, local bays, etc. This subscript must be determined in context.

L Lower bearing or seal

M Main gear

N Nose gear

m A contacting element

o The initial value

p Pad, or piston

S Return stroke

T Tire

U Upper bearing or seal

V Vertical beam

W Gravitational, wind, wing

x, y, z	Components along the x, y, and z axes; or, in a matrix, referring to the component of displacement in the corresponding kinetic energy term.
x', y', z'	Components along the x', y', and z' axes; or, in a matrix referring to the component of displacement in the corresponding kinetic energy term.

SECTION 1

INTRODUCTION

The procedures accepted by the aircraft industry for determining the design loads for alighting gear were until recent years based on the assumption that an adequate design could be obtained from consideration of a few discrete conditions. These conditions and the procedures for determining the associated loads were established in the specifications. The great increases in airplane performance subsequent to World War II, however, caused the reinvestigation of many well established criteria. The introduction of high speed computing equipment about this same time made possible the solution of problems that previously were too long and complicated to permit economical solution on a large scale.

The aircraft alighting gear was recognized as one design area that could benefit from the performance of more detailed analyses and the rationalization of procedures for determining design loads. The first efforts in this direction led to the consideration of mathematical models which had one or two degrees of rigid body freedom. These efforts eventually led to a multiple degree of freedom analysis based on a rigid airplane free to translate horizontally and vertically and to pitch. The motion of the airplane was defined in the ground reference system. The equations of motion were simple, although non-linear, equations.

Comparison of the numerical data obtained from solving these equations with the data obtained from airplane flight and drop test programs indicated that the equations could be made to yield more accurate answers provided additional physical mechanical details were considered in the analytic formulation. It eventually became apparent that considerable simplicity could be gained by using the body axis system as the primary coordinate system in lieu of the ground reference system. Subsequently, the equations of motion were written in the body reference system and due to the pitch rotation of the airplane a Coriolis force term appeared on the left hand side of the equation.

The addition of other degrees of freedom for motion of the rigid vehicle and the effects of structural deflections and other pertinent considerations caused the continued build-up of the equations of motion and the concomitant force and geometric expressions. By this process, which might be called a building block procedure (that is, a procedure which is built up term for term as the necessity for each term is recognized), the procedure for determining alighting gear loads was extended to a multiple degree of freedom analysis which included not only the six rigid body degrees of freedom, but also contained degrees of freedom for a flexible airframe. The building block approach has provided a procedure that is very adequate for the determination of alighting gear loads for conventional airplanes.

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The advent of flight vehicles that cannot be classed as conventional airplanes has, however, created new problems. It is no longer possible in all cases to define for an unknown configuration the parameters important to the proper solution. Consequently, at the beginning of this program, it was decided that providing a rational method that would fulfill the stated requirements would necessitate a departure from the building block approach. A development program was therefore established on the basis of a completely general approach to the problem.

In an effort to provide a broad base of understanding of the program, and of this report, the salient ideas and concepts involved in the program and the general format of this report, are presented in the following paragraphs. The ideas and concepts will be discussed with the intention of providing the reader with a word picture of the processes being employed. To provide a rational method for predicting alighting gear loads that would attain the wide applicability desired has necessitated the utilization of numerous mathematical concepts and tools. As is frequently the case in mathematical developments, the derivation of a very general result requires greater effort than the development of a specific result.

Since the problem being solved is a complex problem in structural dynamics, it is obvious that the development of the method will require employment of the concepts and tools of dynamics. However, the method is presented in a concise, easy to follow, didactic manner so that the average dynamics engineer can readily apply this method to his specific problem. Consequently, the derivation of the equations of motion are presented under the assumption that the careful reviewer of the mathematical development has a working knowledge of the Lagrangian equations of motion, vector calculus, matrices, and structural analysis methods. The presentation of the procedures involved in the use of the method will not, however, require detailed knowledge of these concepts, and the aim of the presentation will be to facilitate use of the method by the engineer. As a preview to the technical presentation, the highlights of the development of the method are presented here in the introduction.

Several iterations were necessary to the evaluation of the equations of motion in their final form. After the first formulation of the general equations, it became evident that a simplification of expression was necessary to allow the equations to be presented in a more concise manner. The equations were then rewritten in matrix form, which provided a means of collecting terms in a manner that would allow the equations of motion to be viewed as a set of equations rather than as individual equations. As work on the project progressed to the consideration of some of the detailed problems concerned with representation of a complex structure consisting of several components, it was found that the inertial characteristics could be handled more readily if the characteristics of each component were expressed in its own coordinate system. Consequently, the equations governing each component motion relative to the vehicle were expressed explicitly, and the general equations of motion were again rewritten. This form provides generality and also is readily adaptable to numerical analysis.

To provide a mathematical model of the vehicle that is of sufficient generality, the position of every point in the vehicle is defined in terms of the position of a discrete set of points in the vehicle, called panel points. Mass is considered to be distributed throughout the vehicle;

distributed mass inertial characteristics are considered, but concentrated mass points may later be used for a particular model. The panel points are allowed only to translate with respect to each other along three mutually perpendicular axes. The mathematical model consists of N components, such as wing, tails, landing gears, which are elastic when the panel points on a component are allowed relative motions, or which may be made rigid by allowing no relative motions. Rigid body displacement of a set of panel points with respect to a component coordinate system is also allowed in order to account for motions such as those of control surfaces and gear stroking.

With the basic mathematical model thus established it is possible to proceed with the establishment of a workable notation, to define the coordinate systems to be employed, and to establish the procedure required to maintain knowledge of the position of the panel points. The right hand rule is used in defining the coordinate systems. The ground coordinate system is considered to be an inertial frame of reference. The body coordinate system is defined, as are the component coordinate systems, and the transformations from one system to another are established; hence, the wing coordinates of a point on the wing can be transformed to body coordinates or to the inertial frame by these transformations. Maintaining knowledge of the position of the panel points is accomplished by defining position vectors such that the position of every panel point relative to the inertial frame is known.

To provide the desired generality in the most direct manner, the Lagrangian equations of motion are employed as the foundation of the development. The use of the Lagrangian equations requires that the potential and kinetic energy of the system be defined in terms of the ground system coordinates, but in general this is at best a very difficult task. However, if use is made of the body coordinate system, the kinetic and potential energy of the system can be readily defined. It would be desirable, then, to express the Lagrangian equations in terms of body coordinate system variables. This can be done by writing a set of Lagrangian equations to define the translational and rotational motion of the body coordinate system and another set to define the location of all particles in the body with respect to the origin of the body coordinates. The rotational position is defined in terms of Eulerian angles. These angles are then so chosen that a transformation from body coordinates to the inertial frame is always possible. It is therefore possible to transform the inertial velocities to body velocities.

Since the kinetic energy of the body is written in terms of the body velocities, the elastic displacements of a discrete number of panel points from their equilibrium position, and the velocities of these panel points in their components, the requirement is to express the Lagrangian equations in the terms used for the kinetic energy. The development of these equations was performed for this program. The results of transforming the Lagrangian equations are shown as "modified Lagrangian equations". The general equations of motion are then derived by substituting the equations for the kinetic and potential energy into the "modified Lagrangian equations" and performing the indicated operations. The resulting equations are long and contain numerous terms that by inspection can be seen to be of no importance to the alighting loads problem. In the general presentation in Appendix C all terms are shown, but for the purpose of this report, the unimportant terms have been deleted in Section 2, where the resulting equations of motion governing landing impact are shown. Equations 2.8-3 and 2.8-4 are referred to as "rigid body"

equations since the principal terms are the inertial terms of the undeflected vehicle; the remaining terms on the left hand side are inertial coupling terms due to elastic deformations and component motions. The terms on the right hand side of Equations 2.8-3 and 2.8-4 are the external forces and moments, respectively, written in the body coordinate system. Equation 2.8-5 without the "rigid body" coupling terms are the ordinary panel point equations of a restrained body, with the panel point forces and constraint forces appearing on the right hand side of the equation. These three equations must be solved simultaneously with any existing constraint equations in order to obtain a solution.

It is intended that the formulation presented here be of sufficient generality to fulfill the requirements of any particular problem concerning landing impact. It is also intended that the manner of presentation of the method be of such clarity to permit utilization of the method by the practicing engineer.

The requirement for generality has caused some complexity in the formulation, which has resulted in a considerable amount of discussion to explain fully the myriad of details covered by the program. The intent of this lengthy discussion is to allow the user of the method the freedom of choosing the particular segments applicable to his problem. By proper choice of terms from the general expressions, consistent with any simplifying assumptions that have been made, the equations of motion for a particular problem can be obtained from this report. The utility of this procedure depends upon whether or not significant time can be saved in formulating the desired equations of motion by using this report, as contrasted to the time required to formulate the equations independently.

To assist in attaining the desired utility, it has been necessary to temper somewhat the desire for generality of the final expressions describing the mathematical model. Therefore, for the development in the main body of the report certain assumptions have been made which eliminate some terms whose effect on alighting gear loads or vehicle motion is negligible for any known or postulated vehicle configuration.

Section 2 of this report is devoted to the development of the equations of motion. This section presents a somewhat detailed explanation of the various mathematical concepts employed in the development of the method. It has not been possible due to space requirements to include in Section 2 all the algebraic manipulations required in the development. As noted above, certain simplifying assumptions made in Section 2 have resulted in the omission of certain terms in the final equations. For the benefit of those dynamicists who are interested in the complete development of the equations, Appendices A, B, and C are included. In these appendices the entire development is presented without recourse to simplification.

> For those engineers not interested in the development, the final equations of motion, Equation 2.8-3, 2.8-4, and 2.8-5, from Section 2 may be used with the information of the subsequent sections to obtain the desired problem formulation.

Section 3 presents the definitions of and the formulae applicable to the applied loads, which are shown on the right hand side of the equations of

motion (Equations 2.8-3, 2.8-4, and 2.8-5). The importance of this section of the report lies in the procedures for the proper introduction of these forces into the system of equations. The specific information on procedures for calculating applied loads is based on the best currently available data. These procedures, however, may not always exactly fit the physical condition being simulated. It is necessary, then, that the user decide upon the applicability of the given expressions, and, if necessary, modify these expressions or completely define new expressions applicable to the vehicle being investigated. In the latter case, the information of Section 3 becomes the guide to be used in the formulation, and provides the necessary framework for proper introduction of newly defined forces into the equations of motion.

The procedures whereby the results of Sections 2 and 3 may be manipulated for use in the landing analyses of vehicles of current interest are presented in Section 4, Vol. II. It is expected that Section 4, Vol. II should be sufficiently complete to make the report valuable even without a good understanding of the Lagrangian formulation of mechanics. Section 4, Part II explains the proper procedures for utilizing the method.

Vol. II is concluded with the presentation in Section 5 of several numerical solutions of the example problems in Section 4. These problems are intended to exemplify the salient features of the method. To provide ease of understanding, an attempt has been made to present specific aspects of the method in each example so that the user will not be confronted with an excessive number of new concepts in a single problem. After an investigation of the information in the report, (Vol. I and II), the user should be able to proceed to the solution of more detailed and extensive problems.

Providing a complete and rational method for predicting alighting gear loads during landing impact has necessitated the presentation of many diverse but related topics. Some of these are basic to the development; for example, distributed mass and rotational inertias and distributed bending and torsional stiffness of all parts of the vehicle, as represented by the various A and K matrices in the equations, are inherent in the mathematical model used for the development, and thereby vehicle flexibility is included. The exact configuration, shape and dimensions of all basic components of the vehicle being studied, as well as those of the composite vehicle, are required for the purpose of defining the various forces. The mathematical model is sufficiently general to allow use of one, two, three, four or more separate alighting gears for vehicle support. The mathematical model also provides the freedom required for establishment of variations in coefficients of other parameters, and of any desired combinations of inital conditions on displacements, rates, or accelerations; for example, rigid body pitch (attitude) and pitch rate may be specified simply as initial conditions, and the desired initial value of pitch acceleration may be obtained by the proper unbalance of moments in the pitch degree of freedom equation of motion. (This same statement is applicable to vertical, longitudinal and lateral translations and to roll and yaw rotations.)

To provide a handy reference for inital utilization of this report, some of the additional topics, which are not general throughout the report, but which are worthy of special note, are presented in Table 1, page seven. This table shows the page location for some of the general topics as well as for such special topics as friction forces and mechanical springs.

While it is not feasible to cover in such a table all details included in the report, most major special topics are enumerated.

TABLE 1

LOCATION OF TOPICAL DISCUSSIONS

SECTION 2

FORMULATION OF THE EQUATIONS OF MOTION

2.1 GENERAL

The formulation of the equations of motion given in this section, although somewhat formidable algebraically, can be described in simple terms. From the definition of inertial and body coordinate systems the kinetic and potential energies for the mathematical model under consideration are written in the body coordinate system. These energy expressions are then substituted into a modified expression of Lagrange's equations, yielding the equations of motion of the vehicle during landing impact. The desire for generality and compactness in the final equations has dictated that the variables chosen to define the vehicle motion be those expressed in the accelerating, or body, reference system. This choice of variables requires that the operations associated with the usual form of Lagrange's equations be transformed to operations in the accelerating reference frame.

In order that continuity may be maintained, many of the detailed steps of the derivation have been deferred to the appendices.

2.2 MATHEMATICAL MODEL OF THE VEHICLE

The vehicle under consideration is composed of N components, each of which may be elastic. Each component is assigned a subscript i , so that $i = 1, 2, ---, N$. The subscript which is unity refers to the main component or fuselage. The remaining subscripts refer to wings, tails, gears, etc. Each of these components has a defined volume, a mass density, and an initial orientation in space.

The motion of the continuous distribution of mass requires an infinity of variables to be exactly described. It is approximated by the motion of a discrete number of points called panel points. Mass and stiffness properties are assigned to the panel points by some interpolation scheme such that the energies of the continuous system and the panel point model are equivalent.

The motion of the panel points will be defined relative to the body axis, which are not fixed in the inertial frame of reference. These motions must then be described relative to the inertial frame in order to formulate the kinetic energy of the system.

2.3 NOTATION

The notation in this report is made consistent with that in the field of aerodynamics where possible. The size of the report indicates that duplication of symbols may occur. The forms to be used will be consistent throughout the report, and symbols used more than once are defined for each usage.

General notation as used in this report is as follows:

Vectors - Vectors are denoted by a double bar on the left edge of the symbol; $\mathbb{P}$, $\mathbb{R}$, $\mathbb{L}$.

Matrices - Matrices are denoted by a symbol enclosed by braces or brackets, the distinction between types are listed:

$[\ \ ]$ - A rectangular matrix; the number of rows and columns depends on the particular matrix

$[\ \ ]'$ - A rectangular matrix; the transpose of the matrix indicated by the brackets and enclosed symbol

$\{\ \ \}$ - A column matrix; a single column of elements as indicated by an enclosed symbol

$\{\ \ \}'$ - A row matrix; the transpose of the indicated column matrix

$[\diagdown 1 \diagdown]$ - The identity matrix; a matrix with unity in the diagonal positions and zeroes off the diagonal

$\begin{Bmatrix} \{\ \} \\ \{\ \} \end{Bmatrix}, \begin{pmatrix} \{\ \} \\ \{\ \} \end{pmatrix}$ - A column matrix composed of the indicated column matrices

Time derivatives - A partial derivative of a variable with respect to time is indicated by placing a dot above the variable, once for each time it is differentiated;

$$\frac{\partial f}{\partial t} = \dot{f} \quad , \quad \frac{\partial^2 f}{\partial t^2} = \ddot{f}$$

Primes - When placed on a matrix, the matrix transpose is indicated. When placed on a variable or subscript to a variable, the quantity is considered to be written in a component coordinate system.

Particular cases of the matrix notation involve combinations of the above notation. These cases are consistent in notation and generally will not be explained in detail.

Symbols and notation of components of the vehicle are considered in the definition of coordinate systems and vectors.

2.4 COORDINATE SYSTEMS

Three distinct coordinate systems, inertial, body, and component, are used to define the motion. Each of the coordinate systems is defined by a right handed triad of mutually orthogonal unit vectors which specify the directions of the coordinate axes.

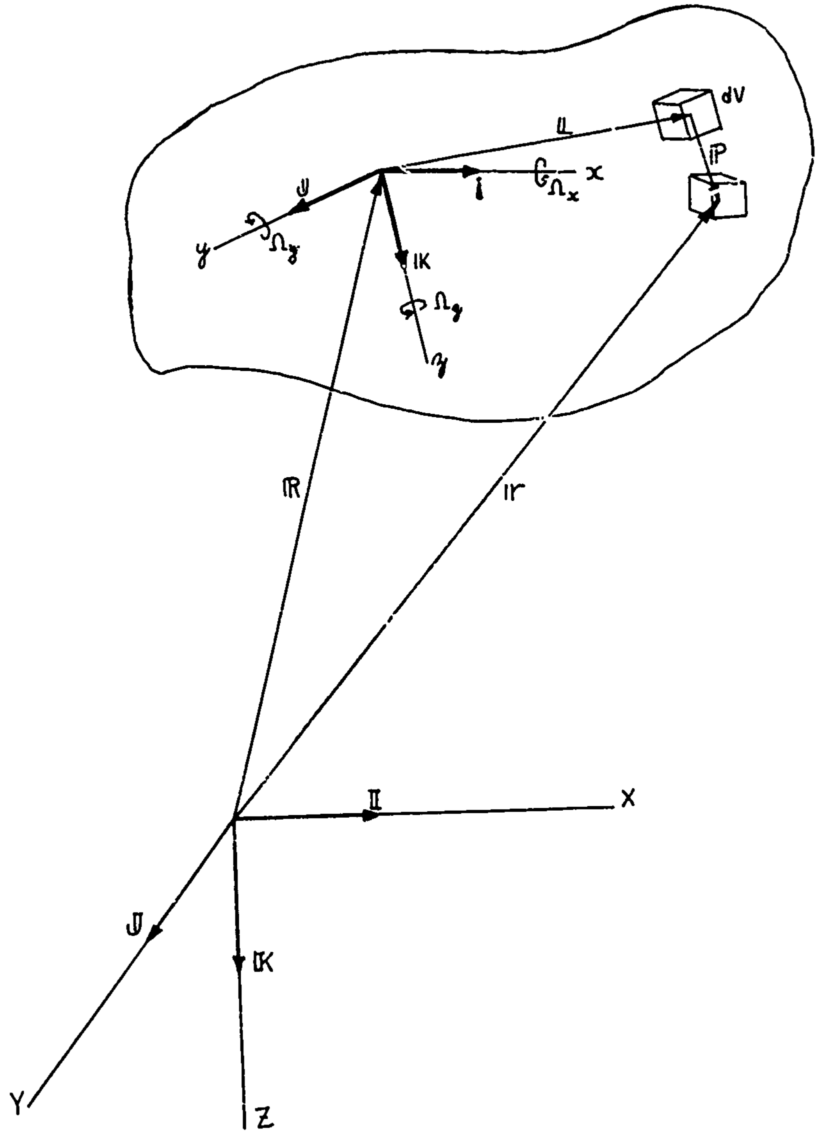


Figure 1. Position Vectors

The inertial frame or ground frame of reference is specified by the unit vectors $\mathbb{I}$, $\mathbb{J}$, $\mathbb{K}$. The unit vector $\mathbb{K}$ is normal to the horizontal ground plane and is positive downward. The unit vectors $\mathbb{I}$, $\mathbb{J}$ are in the ground plane.

The principal or body coordinate system is specified by the unit vectors $\mathbb{i}$, $\mathbb{j}$, $\mathbb{k}$. The orientation of these vectors is specified initially by the principal axes of inertia of the undeflected vehicle, and thereafter by the equations of motion. The unit vector $\mathbb{k}$ is considered positive downward and the unit vector $\mathbb{i}$ positive forward, where those directions have meaning.

Component coordinate systems are specified by the unit vectors $\mathbb{i}'_i$, $\mathbb{j}'_i$, $\mathbb{k}'_i$, The subscript refers to the particular component. The origins of these component coordinate systems are coincident with that of the principal or body coordinate system and are fixed relative to the principal or body coordinate system. These coordinate systems are oriented in each component such that the motion of that component is as simple to describe as possible. This concept is discussed as various component problems arise.

Transformations between the coordinate systems are given by

$$\begin{Bmatrix} \mathbb{i} \\ \mathbb{j} \\ \mathbb{k} \end{Bmatrix} = \left[\Gamma \right] \begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix}$$

Inertial to vehicle body coordinate system

$$\begin{Bmatrix} \mathbb{i} \\ \mathbb{j} \\ \mathbb{k} \end{Bmatrix} = \left[\gamma \right]_i \begin{Bmatrix} \mathbb{i}' \\ \mathbb{j}' \\ \mathbb{k}' \end{Bmatrix}_i$$

Component to vehicle body coordinate system

where the elements of the transformations are the direction cosines between the coordinate axes, given by

$$\left[\Gamma \right] = \begin{bmatrix} \mathbb{i} \cdot \mathbb{I} & \mathbb{i} \cdot \mathbb{J} & \mathbb{i} \cdot \mathbb{K} \\ \mathbb{j} \cdot \mathbb{I} & \mathbb{j} \cdot \mathbb{J} & \mathbb{j} \cdot \mathbb{K} \\ \mathbb{k} \cdot \mathbb{I} & \mathbb{k} \cdot \mathbb{J} & \mathbb{k} \cdot \mathbb{K} \end{bmatrix}$$

$$\left[\gamma \right]_i = \begin{bmatrix} \mathbb{i} \cdot \mathbb{i}'_i & \mathbb{i} \cdot \mathbb{j}'_i & \mathbb{i} \cdot \mathbb{k}'_i \\ \mathbb{j} \cdot \mathbb{i}'_i & \mathbb{j} \cdot \mathbb{j}'_i & \mathbb{j} \cdot \mathbb{k}'_i \\ \mathbb{k} \cdot \mathbb{i}'_i & \mathbb{k} \cdot \mathbb{k}'_i & \mathbb{k} \cdot \mathbb{k}'_i \end{bmatrix}$$

Since this method is primarily concerned with the cantilevered type of landing gears, it is assumed for the general formulation that the component transformation matrices $[\gamma]_i$ are independent of time. This assumption will then allow simple descriptions of component motions if the orientation of the components are fixed relative to the body coordinate system. This simplification keeps the size of the formulation within reason while not compromising the accuracy for the intended application. In Section Four the logical extension of the equations of motion to include articulated landing gears will be demonstrated by example.

The matrix $[\Gamma]$ is the Eulerian transformation from the ground coordinate system to the body coordinate system. It is given by Equation 2.7-3, and is time dependent.

2.5 POSITION AND VELOCITY VECTORS

Any formulation of a dynamics problem requires that the position of all masses be known relative to an inertial frame. With reference to Figure 1, the following vectors must then be defined.

$\mathbb{R}$ - The position vector of the origin of the principal or body coordinate system:

$$\mathbb{R} = X\,\mathbb{I} + Y\,\mathbb{J} + Z\,\mathbb{K}$$

$\dot{\mathbb{R}}$ - The velocity of the origin of the body coordinate system:

$$\dot{\mathbb{R}} = \dot{X}\,\mathbb{I} + \dot{Y}\,\mathbb{J} + \dot{Z}\,\mathbb{K}$$

$$\dot{\mathbb{R}} = v_x\,\mathbb{i} + v_y\,\mathbb{j} + v_z\,\mathbb{k}$$

$\mathbb{L}$ - The position vector from the origin of the body coordinate system to the undeflected position of an elemental volume dV:

$$\mathbb{L} = x\,\mathbb{i} + y\,\mathbb{j} + z\,\mathbb{k}$$

$\mathbb{r}$ - The position vector of an elemental volume dV from the origin of the ground or inertial reference system.

$\boldsymbol{\Omega}$ - The angular velocity of the body expressed in the body coordinate system

$$\boldsymbol{\Omega} = \Omega_x \mathbb{i} + \Omega_y \mathbb{j} + \Omega_z \mathbb{K}$$

$\mathbb{P}$ - The total displacement of an elemental volume relative to the body coordinate system, expressed in the component coordinate system

$$\mathbb{P} = P_{x'} \mathbb{i}' + P_{y'} \mathbb{j}' + P_{z'} \mathbb{K}'$$

In addition, there are several vectors which must be defined for use later in the formulation.

$\boldsymbol{\Delta}'_i$ - The displacement of a volume element dV from its undeflected position defining component rigid body displacement with respect to the body axes, expressed in the component coordinate system in which the motion is most easily described, (referred to as "delta" motion).

$\mathbb{P}_{ij}$ - The displacement of a volume element in the i -th component due to displacement of the j -th component, $\mathbb{P}_j$. The letter j here simply refers to the particular component to which the i -th component is attached.

$\mathbb{P}_i^e$ - The displacement vector of an elemental volume due to elastic deformations of the i -th component only,

$$\mathbb{P}_i^e = P^e_{x'_i} \mathbb{i}' + P^e_{y'_i} \mathbb{j}' + P^e_{z'_i} \mathbb{K}'$$

The total displacement of an elemental volume is then the sum of the latter three vectors.

$$\mathbb{P}_i = \mathbb{P}_i^e + \mathbb{P}_{ij} + \boldsymbol{\Delta}'_i$$

For the main component, or fuselage, there is no "delta" motion or motion caused by other displacements, and

$$\mathbb{P}_1 = \mathbb{P}_1^e$$

The definition of the principal or body coordinate system requires that it be initially the principal axis of inertia system, so that by defintion

$$\int_V \mathbb{L}(x, y, z)\, \rho(x, y, z)\, dV = 0$$

where ρ (x, y, z) is the mass density of the vehicle and V is the total volume.

2.6 KINETIC AND POTENTIAL ENERGY OF THE SYSTEM

The kinetic and potential energy of the system can now be written.

The kinetic energy of any system may be defined by

$$T = \frac{1}{2}\int_V \left(\frac{d\mathbb{r}}{dt}(x,y,z) \right)^2 \rho \, dv \qquad (2.6\text{-}1)$$

where the vector $\mathbb{r}$ is the position vector of an elemental mass relative to an inertial frame of reference. From Figure 1, the position vector is seen to be

$$\mathbb{r} = \mathbb{R} + \mathbb{L} + \mathbb{P} \qquad (2.6\text{-}2)$$

In vector calculus, it is shown that the total time derivative of a vector expressed in an accelerating coordinate system is found from the operation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \Omega \times \qquad (2.6\text{-}3)$$

This theorem together with the definitions of the above vectors will be used to find the velocity of the elemental mass relative to the inertial frame. The vector $\mathbb{R}$ is a vector in the inertial frame of reference, so that

$$\frac{d\mathbb{R}}{dt} = \dot{\mathbb{R}} \qquad (2\text{-}6\text{-}4)$$

The vector $\mathbb{L}$ is expressed in the body coordinate system, so that it is dependent on body orientation, but it is not explicity dependent on time. Then

$$\frac{d\mathbb{L}}{dt} = \Omega \times \mathbb{L} \qquad (2.6\text{-}5)$$

The total displacement vector of an elemental volume is dependent on time and body orientation;

$$\frac{d\mathbb{P}}{dt} = \dot{\mathbb{P}} + \Omega \times \mathbb{P} \qquad (2.6\text{-}6)$$

The velocity of an elemental volume with respect to the ground reference system is then

$$\frac{d\mathbb{r}}{dt} = \dot{\mathbb{R}} + \Omega \times \mathbb{L} + \dot{\mathbb{P}} + \Omega \times \mathbb{P} \qquad (2.6\text{-}7)$$

The square of this quantity is

$$\frac{d\mathbb{r}}{dt}\cdot\frac{d\mathbb{r}}{dt} = \dot{\mathbb{R}}^2 + (\boldsymbol{\Omega}\times\mathbb{L})^2 + \dot{\mathbb{P}}^2 + (\boldsymbol{\Omega}\times\mathbb{P})^2$$
$$+2\,\dot{\mathbb{R}}\cdot\boldsymbol{\Omega}\times\mathbb{P} + 2\,\dot{\mathbb{R}}\cdot\dot{\mathbb{P}} + 2\,\dot{\mathbb{R}}\cdot(\boldsymbol{\Omega}\times\mathbb{L})$$
$$+2\,\boldsymbol{\Omega}\cdot\mathbb{L}\times\dot{\mathbb{P}} + 2\,\boldsymbol{\Omega}\cdot\mathbb{P}\times\dot{\mathbb{P}}$$
$$+2\,(\boldsymbol{\Omega}\times\mathbb{L})\cdot(\boldsymbol{\Omega}\times\mathbb{P}) \qquad (2\text{-}6\text{-}8)$$

This form is substituted into the expression for kinetic energy given by Equation 2.6-1. The volume integration of the above terms containing panel point displacements or velocities is broken into integrations over each volume with the resulting summation indicated.

$$T = \frac{1}{2}\int_V \dot{\mathbb{R}}^2\rho\,dV + \frac{1}{2}\int(\boldsymbol{\Omega}\times\mathbb{L})^2\rho\,dV$$
$$+\sum_{i=1}^{i=N}\Bigg(\frac{1}{2}\int_{V_i}\dot{\mathbb{P}}_i^2\rho\,dV + \frac{1}{2}\int_{V_i}(\boldsymbol{\Omega}\times\mathbb{P}_i)^2\rho\,dV$$
$$+\dot{\mathbb{R}}\cdot\boldsymbol{\Omega}\times\int_{V_i}\mathbb{P}_i\,\rho\,dV + \dot{\mathbb{R}}\cdot\int_{V_i}\dot{\mathbb{P}}_i\,\rho\,dV + \boldsymbol{\Omega}\cdot\int_{V_i}(\mathbb{L}_i\times\dot{\mathbb{P}}_i)\rho\,dV$$
$$+\boldsymbol{\Omega}\cdot\int_{V_i}(\mathbb{P}_i\times\dot{\mathbb{P}}_i)\rho\,dV + \int_{V_i}(\boldsymbol{\Omega}\times\mathbb{L}_i)\cdot(\boldsymbol{\Omega}\times\mathbb{P}_i)\,\rho\,dV\Bigg) \qquad (2.6\text{-}9)$$

The term

$$\dot{\mathbb{R}}\cdot\int_V(\boldsymbol{\Omega}\times\mathbb{L})\,\rho\,dV \qquad (2.6\text{-}10)$$

does not appear due to the definition of $\mathbb{L}$.

The desired form for the kinetic energy is obtained by expanding the vector quantities in Equation 2.6-9 into component form and performing the indicated operations. The integrals are then evaluated to give the form of the kinetic

energy consistent with the mathematical model described in Section 2.2. The detailed manipulations required to arrive at the desired final expression for the kinetic energy, shown in Equation 2.6-11, are outlined in Appendix A.

$$
\begin{aligned}
T = {} & \frac{1}{2} M \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}' \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} \\
& + \sum_{i=1}^{i=N} \left[\frac{1}{2} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}' \begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \right. \\
& + \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \\
& + \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [v][\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \\
& + \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{0\}' & -\{P_{z'}\}'[A_{z'y'}] & \{P_{y'}\}'[A_{y'z'}] \\ \{P_{z'}\}'[A_{z'x'}] & \{0\}' & -\{P_{x'}\}'[A_{x'z'}] \\ -\{P_{y'}\}'[A_{y'x'}] & \{P_{x'}\}'[A_{x'y'}] & \{0\}' \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \\
& + \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{0\}' & -\{1\}'[z'A_{y'}] & \{1\}'[y'A_{z'}] \\ \{1\}'[z'A_{x'}] & \{0\}' & -\{1\}'[x'A_{z'}] \\ -\{1\}'[y'A_{x'}] & \{1\}'[x'A_{y'}] & \{0\}' \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}
\end{aligned}
\quad (2.6\text{-}11)
$$

(Summation continued on next page)

$$
+\frac{1}{2}\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}
(\{P_{y'}\}'[A_{y'y'}]\{P_{y'}\}+\{P_{z'}\}'[A_{z'z'}]\{P_{z'}\}) & -\{P_{x'}\}'[A_{x'y'}]\{P_{y'}\} & -\{P_{x'}\}'[A_{x'z'}]\{P_{z'}\}\\
-\{P_{y'}\}'[A_{y'x'}]\{P_{x'}\} & (\{P_{x'}\}'[A_{x'x'}]\{P_{x'}\}+\{P_{z'}\}'[A_{z'z'}]\{P_{z'}\}) & -\{P_{y'}\}'[A_{y'z'}]\{P_{z'}\}\\
-\{P_{z'}\}'[A_{z'x'}]\{P_{x'}\} & -\{P_{z'}\}'[A_{z'y'}]\{P_{y'}\} & (\{P_{x'}\}'[A_{x'x'}]\{P_{x'}\}+\{P_{y'}\}'[A_{y'y'}]\{P_{y'}\})
\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}
$$

$$
+\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}
(\{1\}'[{}_{z'}A_{z'}]\{P_{z'}\}+\{1\}'[{}_{y'}A_{y'}]\{P_{y'}\}) & -\{1\}'[{}_{y'}A_{x'}]\{P_{x'}\} & -\{1\}'[{}_{z'}A_{x'}]\{P_{x'}\}\\
-\{1\}'[{}_{x'}A_{y'}]\{P_{y'}\} & (\{1\}'[{}_{x'}A_{x'}]\{P_{x'}\}+\{1\}'[{}_{z'}A_{z'}]\{P_{z'}\}) & -\{1\}'[{}_{z'}A_{y'}]\{P_{y'}\}\\
-\{1\}'[{}_{x'}A_{z'}]\{P_{z'}\} & -\{1\}'[{}_{y'}A_{z'}]\{P_{z'}\} & (\{1\}'[{}_{x'}A_{x'}]\{P_{x'}\}+\{1\}'[{}_{y'}A_{y'}]\{P_{y'}\})
\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}\Bigg]_i
\qquad (2.6\text{-}11)
$$

It should be noted that the subscript i does not appear on each term, but rather on the entire bracket of terms to be summed. Thus all primed subscript terms should have the proper component subscript affixed, as should each coordinate transformation $[T]$. The vehicle velocity components are written in the body coordinate system and the terms involving columns of panel point displacements, $\{P\}$, and their time derivatives, are written in the various component coordinate systems, as is evidenced by the primed subscripts. The transformation of the component coordinate terms into the body coordinate system is evident due to the presence of the coordinate transformation matrices. The matrices $[\Omega]$ and $[\omega]$ are defined in Equations 2.7-11, 12.

The matrices $[A]$ with various subscripts are called mass matrices. They have different forms depending on the type of integral which is evaluated. Each arises from some numerical scheme which relates the discrete form of the kinetic energy to that of the continuous physical system, consistent with the definition of the mathematical model.

It is assumed that the potential energy in the vehicle arises only from structural deformations in the linear range. This requires that elastic displacements remain small; i.e., that the relative displacements of adjoining panel points be small compared to the distance between them. The potential energy may be expanded in a Taylor's series in panel point elastic displacements. If it is expanded about a point of minimum potential (the equilibrium position), and the potential at that point is arbitrarily set equal to zero, the constant and linear terms in elastic displacements will be zero. If all terms higher than second order are neglected, a quadratic form for the potential energy results:

$$U = \frac{1}{2} \sum_{i=1}^{i=N} \begin{Bmatrix} \{P^e_{x'}\} \\ \{P^e_{y'}\} \\ \{P^e_{z'}\} \end{Bmatrix}_i' \begin{bmatrix} [K_{x'x'}] & [K_{x'y'}] & [K_{x'z'}] \\ [K_{y'x'}] & [K_{y'y'}] & [K_{y'z'}] \\ [K_{z'x'}] & [K_{z'y'}] & [K_{z'z'}] \end{bmatrix}_i \begin{Bmatrix} \{P^e_{x'}\} \\ \{P^e_{y'}\} \\ \{P^e_{z'}\} \end{Bmatrix}_i \qquad (2.6\text{-}12)$$

The $[K]$ matrices indicated here are called stiffness matrices, and are derived from the geometry and physical characteristics of the various components of the system. They embody not only the distributed stiffness of a given component, but also the restraints due to supporting members between components. Their determination thus involves both a detailed structural analysis and some sort of numerical technique, so that the continuous structure is properly represented. This subject is too broad to be included in this report; the reader unfamiliar with this aspect of the analysis may consult references 2, 5, 18 and 19.

The above forms for the kinetic and potential energy will be substituted into Lagrange's equations, which are to be developed next.

2.7 LAGRANGE'S EQUATIONS

A derivation of the basic form for Lagrange's equations may be found in reference 1. A summary of the concepts involved in their derivation is necessary in order to understand the contents of this section.

Basic to the derivation is the requirement that the position and orientation of every particle be specified relative to a fixed frame of reference. This is accomplished in terms of any set of variables; as these may be distances, velocities, angles, or perhaps less familiar quantities, they are referred to as generalized coordinates. The appearance of Lagrange's equations varies among authors. A convenient form for use here is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i + \sum_j \sigma_j \frac{\partial F_j}{\partial q_i} \qquad (2.7\text{-}1) \quad i = 1, 2, \cdots M_2$$

The quantity L is known as the Lagrangian, the difference between the kinetic and potential energy: L = T - U. Here the q_i are the generalized coordinates. The minimum number of coordinates which may be chosen is dictated by the number of degrees of freedom allowed in the motion, and the number of constraints imposed. The quantities Q_i are the associated generalized forces; they may actually be forces, moments, etc. These generalized forces are assumed known in terms of the generalized coordinates. The quantities F_j are a set of constraint relations among the generalized coordinates. If there are $M_2 - M_1$ of these constraints, where M_2 is the number of generalized coordinates, then there are M_1 degrees of freedom for the system. The Lagrangian undetermined multipliers, σ_j, are the proper functions to cause each term on the right hand side of the equations to be a constraint force which is consistent with the constraint relationships F_j. As there are M_2 generalized coordinates and $M_2 - M_1$ Lagrange multipliers, and only M_2 Lagrange equations, it is necessary to solve the set of $M_2 - M_1$ constraint relations simultaneously with Lagrange's equations. These will be discussed later.

For the sake of algebraic simplicity, the kinetic energy of the system was written in terms of the body coordinate system variables. Since this system moves with the body, it does not specify the position and spatial orientation of every particle in the body, contrary to the assumptions from which the above statement of Lagrange's equations was derived. It is then necessary to modify either the expression for the kinetic energy so as to be applicable to Lagrange's equations as expressed, or to modify Lagrange's equations to utilize the kinetic energy in its present form.

An investigation of these alternatives has shown that much of the cumbersome algebra can be avoided if Lagrange's equations are modified to accept the kinetic energy in its present form.

Before proceeding, the Eulerian transformation between the ground coordinate system and the body coordinate system will be established. The transformation was defined such that

$$\begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix} = [\Gamma] \begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix} \qquad (2.7\text{-}2)$$

The form for the individual elements in the transformation matrix will depend on the definitions of the Euler angles to be used, that is, the sequence of rotations which take the vehicle from the orientation of the ground reference system to its instantaneous space orientation. The choice of these rotations is arbitrary. A convenient sequence of rotations for landing problems is defined as follows:

(1) Rotate the system through the angle ψ about the Z axis in the right-handed manner (X into Y) to produce the system (X_1, Y_1, Z_1).

(2) Rotate the system (X_1, Y_1, Z_1) through the angle θ about the Y_1 axis in the right-handed manner (Z_1 into X_1) to produce the system (X_2, Y_2, Z_2).

(3) Rotate the system (X_2, Y_2, Z_2) through the angle ϕ about the X_2 axis in the right-handed manner (Y_2 into Z_2) to produce the system (x, y, z).

The coordinate transformation is given then by

$$[\Gamma] = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi & \cos\psi\cos\phi & \cos\theta\sin\phi \\ \quad -\sin\psi\cos\phi & \quad +\sin\psi\sin\phi\sin\theta & \\ \cos\psi\sin\theta\cos\phi & \sin\psi\sin\theta\cos\phi & \cos\theta\cos\phi \\ \quad +\sin\psi\sin\phi & \quad -\cos\psi\sin\phi & \end{bmatrix} \qquad (2.7\text{-}3)$$

The transformation of the body velocity as expressed in the ground reference coordinate system to the expression in the body coordinate system is given by

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = [\Gamma] \begin{Bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{Bmatrix} \qquad (2.7\text{-}4)$$

The transformation from Eulerian angle time derivatives to angular velocities about the body coordinate axes is given by

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = [R] \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \qquad (2.7\text{-}5)$$

where the transformation matrix is

$$[R] = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \qquad (2.7\text{-}6)$$

The particular property of this choice of Eulerian angles useful in landing problems is the existence of the inverse of the matrix $[R]$ when all of the Euler angles are zero. Conventional airplane axes are thus chosen to represent the body coordinate system, and the vehicle orientation is specified.

A set of generalized coordinates suitable to completely specify the motion could be the Euler angles, the components of the position vector to the origin of the body coordinate system, and the panel point displacements. The Lagrangian then would be given by

$$L = L(X, Y, Z, \psi, \theta, \phi, \{P_x'\}, \{P_y'\}, \{P_z'\}) \qquad (2.7\text{-}7)$$

The statement of Lagrange's equations using these generalized coordinates follows immediately from the general form of Equation 2.7-1. Thus

$$\frac{d}{dt}\begin{Bmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \\ \frac{\partial L}{\partial \dot{Z}} \end{Bmatrix} - \begin{Bmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \\ \frac{\partial L}{\partial Z} \end{Bmatrix} = \begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} + \sum_j \sigma_j \begin{Bmatrix} \frac{\partial F_j}{\partial X} \\ \frac{\partial F_j}{\partial Y} \\ \frac{\partial F_j}{\partial Z} \end{Bmatrix} \qquad (2.7\text{-}8)$$

$$\frac{d}{dt}\begin{Bmatrix} \frac{\partial L}{\partial \dot{\psi}} \\ \frac{\partial L}{\partial \dot{\theta}} \\ \frac{\partial L}{\partial \dot{\phi}} \end{Bmatrix} - \begin{Bmatrix} \frac{\partial L}{\partial \psi} \\ \frac{\partial L}{\partial \theta} \\ \frac{\partial L}{\partial \phi} \end{Bmatrix} = \begin{Bmatrix} N_\psi \\ N_\theta \\ N_\phi \end{Bmatrix} + \sum_j \sigma_j \begin{Bmatrix} \frac{\partial F_j}{\partial \psi} \\ \frac{\partial F_j}{\partial \theta} \\ \frac{\partial F_j}{\partial \phi} \end{Bmatrix} \qquad (2.7\text{-}9)$$

$$\left(\frac{d}{dt}\begin{Bmatrix} \left\{\frac{\partial L}{\partial \dot{P}_x'}\right\} \\ \left\{\frac{\partial L}{\partial \dot{P}_y'}\right\} \\ \left\{\frac{\partial L}{\partial \dot{P}_z'}\right\} \end{Bmatrix} - \begin{Bmatrix} \left\{\frac{\partial L}{\partial P_x'}\right\} \\ \left\{\frac{\partial L}{\partial P_y'}\right\} \\ \left\{\frac{\partial L}{\partial P_z'}\right\} \end{Bmatrix} = \begin{Bmatrix} \{Q_x'\} \\ \{Q_y'\} \\ \{Q_z'\} \end{Bmatrix} + \sum_j \sigma_j \begin{Bmatrix} \left\{\frac{\partial F_j}{\partial P_x'}\right\} \\ \left\{\frac{\partial F_j}{\partial P_y'}\right\} \\ \left\{\frac{\partial F_j}{\partial P_z'}\right\} \end{Bmatrix} \right)_i \qquad (2.7\text{-}10)$$

where the last set of equations are written once for each component.

The kinetic energy could be transformed into an expression relating these variables, and the indicated operations performed. The algebra involved in obtaining the desired result, however, may be lessened considerably by transforming the operations in these equations to operations on the variables in which the kinetic energy is already expressed. The derivation of Lagrange's equations in "modified form is shown in detail in Appendix B. In terms of the matrices

$$[\Omega] = \begin{bmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{bmatrix} \qquad (2.7\text{-}11)$$

$$[v] = \begin{bmatrix} 0 & v_z & -v_y \\ -v_z & 0 & v_x \\ v_y & -v_x & 0 \end{bmatrix} \qquad (2.7\text{-}12)$$

the modified statement of Lagrange's equations is shown to be

$$\left([1]\frac{d}{dt} - [\Omega]\right)\begin{Bmatrix} \frac{\partial T}{\partial v_x} \\ \frac{\partial T}{\partial v_y} \\ \frac{\partial T}{\partial v_z} \end{Bmatrix} = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} \qquad (2.7\text{-}13)$$

$$\left([1]\frac{d}{dt} - [\Omega]\right)\begin{Bmatrix} \frac{\partial T}{\partial \Omega_x} \\ \frac{\partial T}{\partial \Omega_y} \\ \frac{\partial T}{\partial \Omega_z} \end{Bmatrix} - [v]\begin{Bmatrix} \frac{\partial T}{\partial v_x} \\ \frac{\partial T}{\partial v_y} \\ \frac{\partial T}{\partial v_z} \end{Bmatrix} = \begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix} \qquad (2.7\text{-}14)$$

$$\frac{d}{dt}\begin{Bmatrix} \left\{\frac{\partial L}{\partial \dot{P}_{x'}}\right\} \\ \left\{\frac{\partial L}{\partial \dot{P}_{y'}}\right\} \\ \left\{\frac{\partial L}{\partial \dot{P}_{z'}}\right\} \end{Bmatrix}_i - \begin{Bmatrix} \left\{\frac{\partial L}{\partial P_{x'}}\right\} \\ \left\{\frac{\partial L}{\partial P_{y'}}\right\} \\ \left\{\frac{\partial L}{\partial P_{z'}}\right\} \end{Bmatrix}_i = \begin{Bmatrix} \{Q_{x'}\} \\ \{Q_{y'}\} \\ \{Q_{z'}\} \end{Bmatrix}_i + \sum_j \sigma_j \begin{Bmatrix} \left\{\frac{\partial F_j}{\partial P_{x'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{y'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{z'}}\right\} \end{Bmatrix}_i \qquad (2.7\text{-}15)$$

$i = 1, 2, \cdots, N.$

2.8 THE EQUATIONS OF MOTION

The desired equations of motion may now be obtained by the substitution of the forms for kinetic and potential energy into the set of Lagrange Equations 2.7-13, 14, 15. The operations indicated there are somewhat lengthy, and are relegated to Appendix C.

The final equations of motion are presented as three sets of equations, each defining one of the vectors describing the motion. The components of the linear velocity vector $\mathbb{R}$ are governed by Equation 2.8-3, and those of the angular velocity Ω by Equation 2.8-4. The total displacement vector $\mathbb{P}$ is governed by Equation 2.8-5. These equations describe the motion relative to the body coordinate axes, which constitute an accelerating frame of reference. The motion relative to the inertial or fixed frame of reference is obtained by transformation of the body velocities, making use of Equations 2.7-4, 5. These velocities are then integrated with respect to time, yielding the components of the position vector $\mathbb{R}$ which specifies the position of the origin of the body coordinate system;

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} - \begin{Bmatrix} X_0 \\ Y_0 \\ Z_0 \end{Bmatrix} = \int_0^t [\Gamma]' \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} dt \,, \tag{2.8-1}$$

and the Eulerian angles which specify the orientation of the body coordinate system by Equation 2.7-2

$$\begin{Bmatrix} \Psi \\ \Theta \\ \phi \end{Bmatrix} - \begin{Bmatrix} \Psi_0 \\ \Theta_0 \\ \phi_0 \end{Bmatrix} = \int_0^t [R]^{-1} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} dt \,. \tag{2.8-2}$$

Up to this point, the number of restrictive assumptions imposed on the mathematical model have been small. The result is that a large number of terms appear in the equations of motion in the appendix. In the interest of presenting a relatively compact set of equations here, some of the terms are deleted from the

Rigid Body Linear Equations

$$M\begin{Bmatrix}\dot{v}_x\\ \dot{v}_y\\ \dot{v}_z\end{Bmatrix} - M[\Omega]\begin{Bmatrix}v_x\\ v_y\\ v_z\end{Bmatrix} + \sum_{i=1}^{i=N}\Bigg[[\gamma]\begin{Bmatrix}\{1\}'[A_{x'}]\{\ddot{P}_{x'}\}\\ \{1\}'[A_{y'}]\{\ddot{P}_{y'}\}\\ \{1\}'[A_{z'}]\{\ddot{P}_{z'}\}\end{Bmatrix}$$

$$-2[\Omega][\gamma]\begin{Bmatrix}\{1\}'[A_{x'}]\{\dot{P}_{x'}\}\\ \{1\}'[A_{y'}]\{\dot{P}_{y'}\}\\ \{1\}'[A_{z'}]\{\dot{P}_{z'}\}\end{Bmatrix} + \left([\Omega][\Omega]-[\dot{\Omega}]\right)[\gamma]\begin{Bmatrix}\{1\}'[A_{x'}]\{P_{x'}\}\\ \{1\}'[A_{y'}]\{P_{y'}\}\\ \{1\}'[A_{z'}]\{P_{z'}\}\end{Bmatrix}\Bigg]_i = \begin{Bmatrix}Q_x\\ Q_y\\ Q_z\end{Bmatrix} \quad (2.8\text{-}3)$$

Rigid Body Angular Equations

$$\begin{Bmatrix}I_{xx}\dot{\Omega}_x\\ I_{yy}\dot{\Omega}_y\\ I_{zz}\dot{\Omega}_z\end{Bmatrix} - [\Omega]\begin{Bmatrix}I_{xx}\Omega_x\\ I_{yy}\Omega_y\\ I_{zz}\Omega_z\end{Bmatrix} + \sum_{i=1}^{i=N}\Bigg[\left([\dot{v}]+[v][\Omega]-[\Omega][v]\right)[\gamma]\begin{Bmatrix}\{1\}'[A_{x'}]\{P_{x'}\}\\ \{1\}'[A_{y'}]\{P_{y'}\}\\ \{1\}'[A_{z'}]\{P_{z'}\}\end{Bmatrix}$$

$$+[\gamma]\begin{bmatrix}\{0\} & \{z'\} & -\{y'\}\\ -\{z'\} & \{0\} & \{x'\}\\ \{y'\} & -\{x'\} & \{0\}\end{bmatrix}'\begin{Bmatrix}[A_{x'}]\{\ddot{P}_{x'}\}\\ [A_{y'}]\{\ddot{P}_{y'}\}\\ [A_{z'}]\{\ddot{P}_{z'}\}\end{Bmatrix}\Bigg]_i = \begin{Bmatrix}N_x\\ N_y\\ N_z\end{Bmatrix} \quad (2.8\text{-}4)$$

Flexible Body (Panel Point) Equations

$$
\begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix}
\begin{Bmatrix} \{\ddot{P}_{x'}\} \\ \{\ddot{P}_{y'}\} \\ \{\ddot{P}_{z'}\} \end{Bmatrix}
+
\begin{bmatrix} [\,0\,] & -2\Omega_{z'}[A_{x'y'}] & 2\Omega_{y'}[A_{x'z'}] \\ 2\Omega_{z'}[A_{y'x'}] & [\,0\,] & -2\Omega_{x'}[A_{y'z'}] \\ -2\Omega_{y'}[A_{z'x'}] & 2\Omega_{x'}[A_{z'y'}] & [\,0\,] \end{bmatrix}
\begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}
$$

$$
+\begin{bmatrix} (\dot{v}_{x'}+\Omega_{y'}v_{z'}-\Omega_{z'}v_{y'})[A_{x'}]' & & \\ & (\dot{v}_{y'}+\Omega_{z'}v_{x'}-\Omega_{x'}v_{z'})[A_{y'}]' & \\ & & (\dot{v}_{z'}+\Omega_{x'}v_{y'}-\Omega_{y'}v_{x'})[A_{z'}]' \end{bmatrix}
\begin{Bmatrix} \{1\} \\ \{1\} \\ \{1\} \end{Bmatrix}
$$

$$
+\begin{bmatrix} -(\Omega_{y'}^2+\Omega_{z'}^2)[A_{x'x'}] & (-\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{x'y'}] & (\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{x'z'}] \\ (\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{y'x'}] & -(\Omega_{x'}^2+\Omega_{z'}^2)[A_{y'y'}] & (-\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{y'z'}] \\ (-\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{z'x'}] & (\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{z'y'}] & -(\Omega_{x'}^2+\Omega_{y'}^2)[A_{z'z'}] \end{bmatrix}
\begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix}
$$

$$
+\begin{bmatrix} -(\Omega_{y'}^2+\Omega_{z'}^2)[A_{x'}]' & (-\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{x'}]' & (\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{x'}]' \\ (\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{y'}]' & -(\Omega_{x'}^2+\Omega_{z'}^2)[A_{y'}]' & (-\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{y'}]' \\ (-\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{z'}]' & (\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{z'}]' & -(\Omega_{x'}^2+\Omega_{y'}^2)[A_{z'}]' \end{bmatrix}
\begin{Bmatrix} \{x'\} \\ \{y'\} \\ \{z'\} \end{Bmatrix}
$$

$$
+\begin{bmatrix} [K_{x'x'}] & & \\ & [K_{y'y'}] & \\ & & [K_{z'z'}] \end{bmatrix}
\begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix}
=
\begin{Bmatrix} \{Q_{x'}\} \\ \{Q_{y'}\} \\ \{Q_{z'}\} \end{Bmatrix}
+\sum_{j} \sigma_j
\begin{bmatrix} \left\{\frac{\partial F_j}{\partial P_{x'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{y'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{z'}}\right\} \end{bmatrix}_i
\qquad (2.8\text{-}5)
$$

$$i = 1, 2, \cdots, N$$

complete set of equations in Appendix C. The Equations 2.8-3, 5 are identical with the expressions obtained in the appendix. The set of Equations 2.8-4 governing the body angular velocity are obtained from Equation C-10 by omitting a number of terms. The omitted terms are of two types. It will be recalled that the body coordinate axes are initially the principal axes of inertia for the undeflected body. If this remained true of the coordinate axes as the panel points move relative to one another, the moments of inertia of the vehicle would still change due to the variation in position of the various distributed masses. These moments of inertia, which are dependent on the panel point motion, are defined by the inertia matrix in Equation C-9, and the manner in which this matrix enters the equation of motion is indicated in Equation C-10. It will be assumed here that the terms involving panel point motions in the inertia matrix may be neglected in comparison to the constant term. In addition, since the body coordinate axes are not instantaneously the principal axes of inertia, the inertial forces associated with the panel point motions will exert a net torque about the body coordinate axes. Of the seven types of terms which arise because of this effect, only the first and third terms listed in Equation C-10 will be retained here.

The deletion of the above terms is justified provided that the principal axes of inertia do not differ appreciably from the body coordinate axes throughout the analysis and that the moments of inertia remain nearly constant. For landing analyses of conventional vehicles this assumption is always true. Configurations which do not satisfy these assumptions will require the additional terms to be retained.

To be complete, the system of equations governing the vehicle motion must include the equations of constraint. These equations are written in the form

$$\{F_j\} = \{0\} \tag{2.8-6}$$

These constraint equations will arise if the set of panel point displacements used to define the motion are not independent variables. If this occurs, Equations 2.8-3,4,5 do not provide enough relations to solve the problem. The remaining relations necessary are the set of constraint equations which state the dependence among the variables. The various constraints that arise in the class of problems considered in this report are discussed in Section 3.5.

<u>Equations defining the rigid body motion of components with respect to the body coordinate axes (Δ - motion).</u>

Although Equations 2.8-3,4,5,6 are a proper description of the motion, they are not in the most convenient form for general application. It will be recalled that the total displacement $\mathbb{P}$ of an incremental volume dV was composed of three displacements:

- $\mathbb{P}_i^e$ - The displacement of an elemental volume due to elastic deformations of the i -th component only.
- $\mathbb{P}_{ij}$ - The displacement of an elemental volume in the i -th component due to the displacement of the j -th component, $\mathbb{P}_j$; where the subscript j refers to the component to which the i -th component is attached.

Δ'_i - The displacement of a volume element dV from its undeflected position defining component rigid body displacement with respect to the body coordinate axes expressed in the component coordinate system.

The displacements $\mathbb{P}^e_i$ and Δ'_i define displacements for separate degrees of freedom; they are independent variables. Thus, if the components of both of these displacements along one of the coordinate axes are retained, there will result more variables than equations. The resulting restriction is therefore made that both of these contributions cannot occur for a given component in the same direction. This will not constitute any real restriction on the use of the equations for the description of landing impact, since the contribution to gear loads from elasticity in the direction in which a component may move relative to the vehicle as a rigid body is small.

The panel point coupling terms in the rigid body equations required consideration of the total displacement $\mathbb{P}$; this requirement and that of compactness dictated the choice of the total panel point displacements as the generalized coordinates used to define the motion. In a particular application, however, the total displacement must be broken into its separate parts. This paragraph will derive explicitly the equations governing Δ -motion for two cases generally required in landing impact analysis; motion along a line such as gear stroking, and motion about a line such as rotation of a bogie about an axle or wheel spinup. Other displacements are discussed in Section 2.9.

In general, the equations describing Δ -motion of a component may be derived from the panel point equations 2.8-5 directly. They are obtained by expanding the total displacement into its separate parts, and setting to zero the elastic displacements of the component in the direction in which component rigid body motion is to be allowed. All terms are retained, however, due to elastic displacements in the remaining direction(s) and displacements due to displacements in another component. For convenience, these inertial coupling terms on the component rigid body motion will be included with the applied panel point forces $\{Q\}$ until the algebraic manipulations are completed. Thus $\{\hat{Q}\}$ will refer to both applied and coupled inertial forces.

Component rigid body motion along a line

The component coordinate system for the component which may move as a rigid body along a line is defined such that one of the axes lies along the line of motion. This will be designated as the $\mathfrak{z}'$-axis. For that component there can be no $\{P^e_{\mathfrak{z}'}\}_i$ according to the previously mentioned restriction. Also, since motion along the $\mathfrak{z}'$ -axis of the member holding this component does not cause motion of the component, then $\{P_{\mathfrak{z}'}\}_{ij}$ is zero, where j refers to the supporting member. Then

$$\{P_{\mathfrak{z}'}\}_i = \{\Delta_{\mathfrak{z}'}\}_i \qquad (2.8\text{-}7)$$

The component is not allowed Δ -motion along the other axes, hence

$$\{P_{x'}\}_i = \{P_{x'}^e\}_i + \{P_{x'}\}_{ij} \tag{2.8-8}$$

$$\{P_{y'}\}_i = \{P_{y'}^e\}_i + \{P_{y'}\}_{ij} \tag{2.8-9}$$

Thus, the component may have elastic deflections along the other axes, and may move in those directions due to motion of the supporting component. The equation for the Δ -motion along a line is then derived from the portion of the panel point equations governing motion along the z' -axis of the i -th component. Use of Equations 2.8-7, 8, 9 yields

$$\left[[A_{z'z'}]\{\ddot{\Delta}_{z'}\} = \{\hat{Q}_{z'}\} + \sum_j \sigma_j \left\{\frac{\partial F_j}{\partial \Delta_{z'}}\right\}\right]_i \tag{2.8-10}$$

The Δ -motion of each panel point is the same for motion along a line, and the column $\{\Delta_{z'}\}$ may be written in terms of any one of the (n) panel point displacements; say the first, $\Delta_{z'_1}$

$$\{\Delta_{z'}\} = \Delta_{z'_1}\{1\} \tag{2.8-11}$$

This may be rewritten as a set of constraint relations

$$\{\Delta_{z'}\} - \Delta_{z'_1}\{1\} = \{0\} \tag{2.8-12}$$

The only contributions to constraints on Δ -motion along a line are then those which make the component move as a rigid body, that is,

$$\{F_j\} = \{\Delta_{z'}\} - \Delta_{z'_1}\{1\} = \{0\} \tag{2.8-13}$$

The constraint term may be calculated immediately;

$$\sum_j \sigma_j \left\{\frac{\partial F_j}{\partial \Delta_{z'}}\right\} = \begin{Bmatrix} -(\sigma_2 + \sigma_3 + \cdots + \sigma_n) \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_n \end{Bmatrix} \tag{2.8-14}$$

The equation defining the motion may be summed by premultiplication by a row matrix of ones, $\{1\}'$

$$\{1\}'[A_{z'z'}]\{1\}\ \ddot{\Delta}_{z'} = \{1\}'\{\hat{Q}\} \tag{2.8-15}$$

The internal constraint forces thus have no net effect on the rigid body motion of the whole component, as expected. Identifying the component mass as

$$M_i = \{1\}'[A_{z'z'}]\{1\} \tag{2.8-16}$$

the mass moments relative to the origin of the body coordinate system are

$$M_i \bar{x}' = \{1\}'[A_{z'}]'\{x'\}$$

$$M_i \bar{y}' = \{1\}'[A_{z'}]'\{y'\}$$

$$M_i \bar{z}' = \{1\}'[A_{z'}]'\{z'\} \tag{2.8-17}$$

Using Equation 2.8-5 and expanding the total displacements in the desired directions under consideration, and setting to zero the elastic displacement of the component in the direction in which component rigid body motion is allowed, the final equation governing Δ -motion along a line is

$$\begin{aligned} M_i \Big[& \ddot{\Delta}_{z'} + \dot{v}_{z'} + \Omega_{x'} v_{y'} - \Omega_{y'} v_{x'} - (\Omega_{x'}^2 + \Omega_{y'}^2)(\bar{z}' + \Delta_{z'}) \\ & + (-\dot{\Omega}_{y'} + \Omega_{x'}\Omega_{z'})\Big(\bar{x}' + \frac{1}{M_i}\{1\}'[A_{z'x'}](\{P_{x'}^{e}\} + \{P_{x'}\}_{ij})\Big) \\ & + (\dot{\Omega}_{x'} + \Omega_{y'}\Omega_{z'})\Big(\bar{y}' + \frac{1}{M_i}\{1\}'[A_{z'y'}](\{P_{y'}^{e}\} + \{P_{y'}\}_{ij})\Big) \\ & - \frac{2\Omega_{y'}}{M_i}\{1\}'[A_{z'x'}]\{\dot{P}_{x'}\}_{ij} + \frac{2\Omega_{x'}}{M_i}\{1\}'[A_{z'y'}]\{\dot{P}_{y'}\}_{ij}\Big] = \{1\}'\{Q_{z'}\} \end{aligned} \tag{2.8-18}$$

The right-hand side is the summation of all the external forces on the component along the z'-axis. The parameters $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ are the coordinates of the component center of mass in its undeflected position. The left side of the equation is the product of the component mass with its instantaneous acceleration along the z'-axis relative to the fixed or ground frame of reference. In most cases, when the component mass is small compared to the total mass, only the first significant terms of the equation need be retained. The remaining terms are small.

<u>Component rigid body motion about a line</u>

The motion of a component about a line as a rigid body is a bit more difficult to forumlate than that along a line. For that reason, the details of the formulation will be presented. This problem is an excellent example, for the interested reader, in the use of constraints.

The coordinate system for Δ-motion about a line will be defined so that the y'-axis of the component system is parallel to the axis of rotation; the motion is then in the x'-z' plane. This implies that $\{\Delta_{y'}\}=\{0\}$. A single row of panel points is laid along a line in the component, one of which is pictured in Figure 2. The angle η measured in the right-handed sense from the x'-axis defines the position of the line of panel points. The distances along the line from the axis of rotation to each panel point are arranged as a column matrix $\{\ell\}$. The convention will be used that the distance is positive when measured along the end defining the angle η, and negative if along the other end.

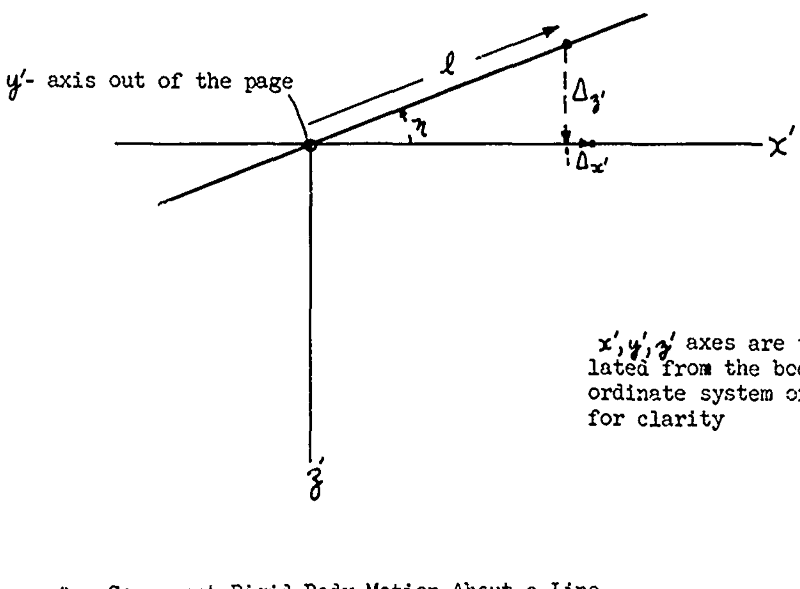


Figure 2. Component Rigid Body Motion About a Line.

From Figure 2, the geometric relations between the angle η and the panel point displacements are seen to be

$$\{\Delta_{x'}\} = \{\ell\}(1 - \cos\eta) \tag{2.8-19}$$

$$\{\Delta_{y'}\} = \{\ell\}\sin\eta \tag{2.8-20}$$

These are a set of equations relating a number of variables, but they are not suitable as constraint equations. Constraint equations can only relate variables already in use. However, the value of any two of the displacements, say $\Delta_{x_1'}$ and $\Delta_{z_1'}$ may be used as

$$\Delta_{x_1'} = \ell_1(1 - \cos\eta) \tag{2.8-21}$$

$$\Delta_{z_1'} = \ell_1 \sin\eta \tag{2.8-22}$$

to write a set of constraints

$$\{\Delta_{x'}\} = \left\{\frac{\ell}{\ell_1}\right\}\Delta_{x_1'} \tag{2.8-23}$$

$$\{\Delta_{z'}\} = \left\{\frac{\ell}{\ell_1}\right\}\Delta_{z_1'} \tag{2.8-24}$$

An additional constraint may be derived from the equations relating $\Delta_{x_1'}$ and $\Delta_{z_1'}$ to the angle by eliminating that angle;

$$\Delta_{x_1'}^{2} - 2\ell_1\Delta_{x_1'} + \Delta_{z_1'}^{2} = 0 \tag{2.8-25}$$

A suitable set of constraints for the motion are then

$$\begin{Bmatrix} F_1 \\ \vdots \\ F_n \end{Bmatrix} = \{\Delta_{x'}\} - \left\{\frac{\ell}{\ell_1}\right\}\Delta_{x_1'} = \{0\} \tag{2.8-26}$$

$$\begin{Bmatrix} F_{n+1} \\ \vdots \\ F_{2n} \end{Bmatrix} = \{\Delta_{z'}\} - \left\{\frac{\ell}{\ell_1}\right\} \Delta_{z_1'} = \{0\} \tag{2.8-27}$$

$$F_{2n+1} = \Delta_{x_1'}^2 - 2\ell_1 \Delta_{x_1'} + \Delta_{z_1'}^2 = 0 \tag{2.8-28}$$

These constraints simply require that all of the panel points rotate together by an amount determined either by Δ_{x_1} or Δ_{z_1}. They may be substituted into the set of equations of motion

$$[A_{x'x'}]\{\ddot{\Delta}_{x'}\} = \{\hat{Q}_{x'}\} + \sum_j \sigma_j \left\{\frac{\partial F_j}{\partial \Delta_{x'}}\right\} \tag{2.8-29}$$

$$[A_{z'z'}]\{\ddot{\Delta}_{z'}\} = \{\hat{Q}_{z'}\} + \sum_j \sigma_j \left\{\frac{\partial F_j}{\partial \Delta_{z'}}\right\} \tag{2.8-30}$$

The constraint terms are

$$\sum_j \sigma_j \left\{\frac{\partial F_j}{\partial \Delta_{x'}}\right\} = \begin{Bmatrix} -\left(\frac{\ell_2}{\ell_1}\sigma_2 + \cdots + \frac{\ell_n}{\ell_1}\sigma_n\right) + \sigma_{2n+1}(2\Delta_{x_1'} - 2\ell_1) \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_n \end{Bmatrix} \tag{2.8-31}$$

$$\sum_j \sigma_j \left\{\frac{\partial F_j}{\partial \Delta_{z'}}\right\} = \begin{Bmatrix} -\left(\frac{\ell_2}{\ell_1}\sigma_{n+2} + \cdots + \frac{\ell_n}{\ell_1}\sigma_{2n}\right) + \sigma_{2n+1}(2\Delta_{z_1'}) \\ \sigma_{n+2} \\ \sigma_{n+3} \\ \vdots \\ \sigma_{2n} \end{Bmatrix} \tag{2.8-32}$$

Substitution of these terms and the constraint relations into the equations of motion yields

$$[A_{x'x'}]\left\{\frac{\ell}{\ell_1}\right\}\ddot{\Delta}_{x'_1} = \{\hat{Q}_{x'}\} + \begin{Bmatrix} -\left(\frac{\ell_2}{\ell_1}\sigma_2 + \cdots + \frac{\ell_n}{\ell_1}\sigma_n\right) + \sigma_{2n+1}(2\Delta_{x'_1} - 2\ell_1) \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_n \end{Bmatrix} \tag{2.8-33}$$

$$[A_{z'z'}]\left\{\frac{\ell}{\ell_1}\right\}\ddot{\Delta}_{z'_1} = \{\hat{Q}_{z'}\} + \begin{Bmatrix} -\left(\frac{\ell_2}{\ell_1}\sigma_{n+2} + \cdots + \frac{\ell_n}{\ell_1}\sigma_{2n}\right) + \sigma_{2n+1}(2\Delta_{z'_1}) \\ \sigma_{n+2} \\ \sigma_{n+3} \\ \vdots \\ \sigma_{2n} \end{Bmatrix} \tag{2.8-34}$$

The Lagrange multipliers may be eliminated immediately. A linear combination of these equations is found by premultiplying the first by $\{\ell/\ell_1\}'\sin\eta$ and the second by $\{\ell/\ell_1\}'\cos\eta$ so that

$$\left\{\frac{\ell}{\ell_1}\right\}'[A_{x'x'}]\left\{\frac{\ell}{\ell_1}\right\}\ddot{\Delta}_{x'_1}\sin\eta = \left\{\frac{\ell}{\ell_1}\right\}'\{\hat{Q}_{x'}\}\sin\eta + \sigma_{2n+1}(2\Delta_{x'_1} - 2\ell_1)\sin\eta \tag{2.8-35}$$

$$\left\{\frac{\ell}{\ell_1}\right\}'[A_{z'z'}]\left\{\frac{\ell}{\ell_1}\right\}\ddot{\Delta}_{z'_1}\cos\eta = \left\{\frac{\ell}{\ell_1}\right\}'\{\hat{Q}_{z'}\}\cos\eta + \sigma_{2n+1}(2\Delta_{z'_1})\cos\eta \tag{2.8-36}$$

If the forms defining $\Delta_{x'_1}$ and $\Delta_{z'_1}$ in terms of η are now substituted into these two equations and they are summed, the last undetermined multiplier is eliminated. The equation then is

$$\begin{aligned}&\{l\}'[A_{x'x'}]\{l\}\left(-\sin\eta\cos\eta\;\dot{\eta}^2-\sin^2\eta\;\ddot{\eta}\right)\\&+\{l\}'[A_{z'z'}]\{l\}\left(\sin\eta\cos\eta\;\dot{\eta}^2-\cos^2\eta\;\ddot{\eta}\right)\\&=\{l\}'\{\hat{Q}_{x'}\}\sin\eta+\{l\}'\{\hat{Q}_{z'}\}\cos\eta\end{aligned}\tag{2.8-37}$$

This may be written in terms of the moment of inertia about the y'-axis as

$$I_A\ddot{\eta}=-\{l\}'\{\hat{Q}_{x'}\}\sin\eta-\{l\}'\{\hat{Q}_{z'}\}\cos\eta\tag{2.8-38}$$

Expanding the inertial terms is a rather long process; the result is

$$\begin{aligned}&I_A\ddot{\eta}+I_A\Big[\left(\Omega_{x'}\cos\eta-\Omega_{z'}\sin\eta\right)\left(\Omega_{x'}\sin\eta+\Omega_{z'}\cos\eta\right)-\dot{\Omega}_{y'}\Big]\\&\quad+M_c\bar{l}\Big[\left(\dot{V}_{x'}\sin\eta+\dot{V}_{z'}\cos\eta\right)+\Omega_{y'}\left(V_{z'}\sin\eta-V_{x'}\cos\eta\right)\\&\qquad+V_{y'}\left(\Omega_{x'}\cos\eta-\Omega_{z'}\sin\eta\right)+\left(\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'}\right)P_{z'_A}\sin\eta\\&\qquad+\ddot{P}_{z'_A}\cos\eta+\left(\Omega_{x'}^2+\Omega_{y'}^2\right)P_{z'_A}\cos\eta\Big]\\&=-\{l\}'\{Q_{x'}\}\sin\eta-\{l\}'\{Q_{z'}\}\cos\eta\end{aligned}\tag{2.8-39}$$

where M_c is the component mass and $\bar{l}$ is the distance from the axle along the line to the center of mass of the component. The inertial moments due to displacement of the axle along the z'-axis, $P_{z'_A}$, are now easily seen. This displacement will be set equal to piston $\Delta_{z'}$ in the case where a bogie element is pivoted on an axle affixed to the end of a gear piston. Moments due to fore and aft and lateral deflections of the axle are neglected.

Particular applications of interest in this report are for bogie motion and wheel spinup. For bogie motion, there are only three panel points at which moment inducing forces are applied; those at the front and rear wheel axles, and one at the point at which the bogie rotational spring-damper is affixed. The distance from the bogie axle to front wheel axle is l_1, that to the rear wheel axle is l_2 (negative), and the bogie equation of motion is

$$I_A \ddot{\eta} + I_A\left[(\Omega_{x'}\cos\eta - \Omega_{z'}\sin\eta)(\Omega_{x'}\sin\eta + \Omega_{z'}\cos\eta) - \dot{\Omega}_{y'}\right]$$
$$+ M_c \ell\left[(\dot{v}_{x'}\sin\eta + \dot{v}_{z'}\cos\eta) + \Omega_{y'}(v_{z'}\sin\eta - v_{x'}\cos\eta)\right.$$
$$+ v_{y'}(\Omega_{x'}\cos\eta - \Omega_{z'}\sin\eta) + (\dot{\Omega}_{y'} + \Omega_{x'}\Omega_{z'})P_{z'_A}\sin\eta$$
$$\left. + \ddot{P}_{z'_A}\cos\eta + (\Omega_{x'}^2 + \Omega_{y'}^2)P_{z'_A}\cos\eta\right] = N(\eta, \dot{\eta})$$
$$-\left[(\ell_1 Q_{x'_1} + \ell_2 Q_{x'_2})\sin\eta + (\ell_1 Q_{z'_1} + \ell_2 Q_{z'_2})\cos\eta\right] \qquad (2.8\text{-}40)$$

where $N(\eta, \dot{\eta})$ is the restraining moment of the rotational spring-damper of the bogie element.

For a round disc-like object rotating through large angles about its central axis, such as a tire and wheel rotating about an axle, the applied forces do not act directly on the line of points which rotates with the wheel. Therefore to define properly the panel point forces would require additional complexity. It is noted, however, that the right hand side of the equations must be the external moment which causes the wheel to rotate, and it can be simply stated as such. The distance ℓ is zero for a balanced wheel, and the inertial terms in body angular motion are small compared to wheel angular acceleration. Hence the equation for wheel angular motion is

$$I_A \ddot{\eta} = \text{EXTERNAL MOMENT} = N_{SU} \qquad (2.8\text{-}41)$$

2.9 TRANSFORMATION TO MODAL COORDINATES

The formulation of the equations of motion as expressed in 2.8-3, 4, 5 was performed with panel point displacements $(\{P_{x'}\}, \{P_{y'}\}, \{P_{z'}\})$ as total displacements relative to the undeflected vehicle. This was necessary for clarity and brevity of the formulation, and does not compromise the generality of the equations. In principle, these equations may be solved to describe the motion during landing of any vehicle. In actual practice, it is desirable to keep the number of variables to a minimum while retaining as high a degree of accuracy as possible. It is not generally desirable to simply reduce the number of panel points until the total number of variables is sufficiently small. A more economic approach is found in the transformation to modal coordinates. The elastic motion of a body may be expressed quite accurately by retaining only those modes whose corresponding frequencies are in the main portion of the frequency spectrum of the forces on the body. Since forces in landing problems are not usually composed of high frequencies, the first few modes are sufficient. However, the mode shape of an entire vehicle is not a concept too useful here, rather, the mode shapes of individual components are considered. The reason for this is fairly simple. If the airplane wing

mass is small compared to the fuselage, the wing elasticity will not contribute to landing loads and may be omitted. One the other hand, the wing mass may be the greatest part, as in a flying delta, and fuselage flexibility may be omitted. The particular problem at hand will then dictate which of the effects to include. With these ideas in mind, it is obvious that the Equation 2.8-5 cannot be transformed into modal coordinates immediately, for the coordinates will depend upon the particular problem.

Mode shapes will be defined for the following set of constraints.

(1) Fuselage modes with constrained rigid body motion, Δ -motion, and elastic motion of all but the fuselage.

(2) Wing modes with constrained rigid body motion, Δ -motion, and elastic motion of all but the wing, unless gears are on the wing.

(3) Modes of all minor elements with all motion constained except the elastic motion of that element, unless another element is attached to that element and not the main body.

Consider the problem of fuselage modes. The displacement of all panel points on the wings may be written directly in terms of displacements of the panel points on the fuselage where the wings are attached. Normally,

$$\begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix} = [T_{WF}] \begin{Bmatrix} \{P_x\} \\ \{P_y\} \\ \{P_z\} \end{Bmatrix}_F + \begin{Bmatrix} \{P^e_{x'}\} \\ \{P^e_{y'}\} \\ \{P^e_{z'}\} \end{Bmatrix}_W + \begin{Bmatrix} \{\Delta_{x'}\} \\ \{\Delta_{y'}\} \\ \{\Delta_{z'}\} \end{Bmatrix}_W \qquad (2.9\text{-}1)$$

Since wing elastic and delta motion are constrained, the constraint equations are

$$\{F_j\} = \begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix}_W - [T_{WF}] \begin{Bmatrix} \{P_x\} \\ \{P_y\} \\ \{P_z\} \end{Bmatrix}_F = \{0\} \qquad (2.9\text{-}2)$$

for each wing. The fuselage panel point equation is

$$\begin{bmatrix} [A_{xx}] & & \\ & [A_{yy}] & \\ & & [A_{zz}] \end{bmatrix}_F \begin{Bmatrix} \{\ddot{P}_x\} \\ \{\ddot{P}_y\} \\ \{\ddot{P}_z\} \end{Bmatrix}_F + \begin{bmatrix} [K_{xx}] & & \\ & [K_{yy}] & \\ & & [K_{zz}] \end{bmatrix} \begin{Bmatrix} \{P_x\} \\ \{P_y\} \\ \{P_z\} \end{Bmatrix}_F = \sum_j \sigma_j \begin{Bmatrix} \left\{\frac{\partial F_j}{\partial P_x}\right\} \\ \left\{\frac{\partial F_j}{\partial P_y}\right\} \\ \left\{\frac{\partial F_j}{\partial P_z}\right\} \end{Bmatrix} \qquad (2.9\text{-}3)$$

and each wing panel point equation is

$$\begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix}_W \begin{Bmatrix} \{\ddot{P}_{x'}\} \\ \{\ddot{P}_{y'}\} \\ \{\ddot{P}_{z'}\} \end{Bmatrix}_W = \sum_j \sigma_j \begin{Bmatrix} \left\{\frac{\partial F_j}{\partial P_{x'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{y'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{z'}}\right\} \end{Bmatrix}_W \qquad (2.9\text{-}4)$$

$$W = W_1 , W_2$$

Suppose that all other masses such as tails, control surfaces, etc. are small. A set of constraints as in equation 2.9-2 is written for both wings; the equations are identical if the wings are mirror images. The right-hand side of 2.9-4 is

$$\sum_j \sigma_j \begin{Bmatrix} \{\partial F_j/\partial P_{x'}\} \\ \{\partial F_j/\partial P_{y'}\} \\ \{\partial F_j/\partial P_{z'}\} \end{Bmatrix}_{W_1} = \{\sigma\}_{W_1} \qquad (2.9\text{-}5)$$

$$\sum_j \sigma_j \begin{Bmatrix} \{\partial F_j/\partial P_{x'}\} \\ \{\partial F_j/\partial P_{y'}\} \\ \{\partial F_j/\partial P_{z'}\} \end{Bmatrix}_{W_2} = \{\sigma\}_{W_2} \qquad (2.9\text{-}6)$$

and the right-hand side of equation 2.9-3 is

$$\sum_j \sigma_j \begin{Bmatrix} \{\partial F_j/\partial P_x\} \\ \{\partial F_j/\partial P_y\} \\ \{\partial F_j/\partial P_z\} \end{Bmatrix}_F = -[T_{WF}]'\left(\{\sigma\}_{W_1} + \{\sigma\}_{W_2}\right) \qquad (2.9\text{-}7)$$

If equations 2.9-5, 6, 7 are substituted into equations 2.9-3,4 and the contraints are used to write wing panel point displacements in terms of fuselage panel point displacements, the equations become

$$\begin{bmatrix}[A_{xx}] & & \\ & [A_{yy}] & \\ & & [A_{zz}]\end{bmatrix}_F \begin{Bmatrix}\{\ddot{P}_x\} \\ \{\ddot{P}_y\} \\ \{\ddot{P}_z\}\end{Bmatrix}_F + \begin{bmatrix}[K_{xx}] & & \\ & [K_{yy}] & \\ & & [K_{zz}]\end{bmatrix}_F \begin{Bmatrix}\{P_x\} \\ \{P_y\} \\ \{P_z\}\end{Bmatrix}_F = -[T_{WF}]'(\{\sigma\}_{W_1} + \{\sigma\}_{W_2}) \tag{2.9-8}$$

$$\begin{bmatrix}[A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}]\end{bmatrix}_{W_1} [T_{WF}] \begin{Bmatrix}\{\ddot{P}_x\} \\ \{\ddot{P}_y\} \\ \{\ddot{P}_z\}\end{Bmatrix}_F = \{\sigma\}_{W_1} \tag{2.9-9}$$

$$\begin{bmatrix}[A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}]\end{bmatrix}_{W_2} [T_{WF}] \begin{Bmatrix}\{\ddot{P}_x\} \\ \{\ddot{P}_y\} \\ \{\ddot{P}_z\}\end{Bmatrix}_F = \{\sigma\}_{W_2} \tag{2.9-10}$$

Premultiplication of equations 2.9-9, 10 by $[T_{WF}]'$ and addition with Equation 2.9-8 eliminate the Lagrange multipliers and yields

$$
\left(\begin{bmatrix} [A_{xx}] & & \\ & [A_{yy}] & \\ & & [A_{zz}] \end{bmatrix}_F + 2[T_{WF}]' \begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix} [T_{WF}] \right) \begin{Bmatrix} \{\ddot{P}_x\} \\ \{\ddot{P}_y\} \\ \{\ddot{P}_z\} \end{Bmatrix}_F \qquad (2.9\text{-}11)
$$

$$
\begin{bmatrix} [K_{xx}] & & \\ & [K_{yy}] & \\ & & [K_{zz}] \end{bmatrix}_F \begin{Bmatrix} \{P_x\} \\ \{P_y\} \\ \{P_z\} \end{Bmatrix}_F = \{0\}
$$

Inclusion of the inertial effects of tail masses, etc. is now obvious. Defining the total body mass matrix by

$$
[A] = \begin{bmatrix} [A_{xx}] & & \\ & [A_{yy}] & \\ & & [A_{zz}] \end{bmatrix}_F + 2\,[T_{WF}]' \begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix} [T_{WF}]
$$

$$
+ 2\,[T_{TF}]' \begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix}_T [T_{TF}] + \text{etc.} \qquad (2.9\text{-}12)
$$

where tails, etc. have been included, the panel point equations defining fuselage modes are

$$
[A] \begin{Bmatrix} \{\ddot{P}_x\} \\ \{\ddot{P}_y\} \\ \{\ddot{P}_z\} \end{Bmatrix}_F + \begin{bmatrix} [K_{xx}] & & \\ & [K_{yy}] & \\ & & [K_{zz}] \end{bmatrix}_F \begin{Bmatrix} \{P_x\} \\ \{P_y\} \\ \{P_z\} \end{Bmatrix}_F = \{0\} \qquad (2.9\text{-}13)
$$

The transformation separating time and space coordinates is

$$
\begin{Bmatrix} \{P_x\} \\ \{P_y\} \\ \{P_z\} \end{Bmatrix}_F = [\phi]_F \begin{Bmatrix} \{q_x\} \\ \{q_y\} \\ \{q_z\} \end{Bmatrix}_F \qquad (2.9\text{-}14)
$$

The fuselage modes are considered to be normalized on the total mass matrix $[A]$, so that

$$\left[\,{}^{\backslash}1_{\backslash}\right]\begin{Bmatrix}\{\ddot{q}_x\}\\ \{\ddot{q}_y\}\\ \{\ddot{q}_z\}\end{Bmatrix}_F + \left[{}^{\backslash}\tfrac{1}{\lambda_F}{}_{\backslash}\right]\begin{Bmatrix}\{q_x\}\\ \{q_y\}\\ \{q_z\}\end{Bmatrix}_F = \{0\} \tag{2.9-15}$$

and

$$[A]\{\phi_F\}_i - \lambda_i^2[K]_F\{\phi_F\}_i = \{0\} \tag{2.9-16}$$

where

$$[\phi]_F = \left[\{\phi_F\}_1, \{\phi_F\}_2, \cdots, \{\phi_F\}_n\right] \tag{2.9-17}$$

define the mode shapes and eigenvalues for the fuselage.

Wing modes are defined with everything constrained except elastic wing motion. Equations 2.8-5 for the wing become

$$\begin{bmatrix}[A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}]\end{bmatrix}_W\begin{Bmatrix}\{\ddot{P}_{x'}^e\}\\ \{\ddot{P}_{y'}^e\}\\ \{\ddot{P}_{z'}^e\}\end{Bmatrix}_W + \begin{bmatrix}[K_{x'x'}] & & \\ & [K_{y'y'}] & \\ & & [K_{z'z'}]\end{bmatrix}_W\begin{Bmatrix}\{P_{x'}^e\}\\ \{P_{y'}^e\}\\ \{P_{z'}^e\}\end{Bmatrix}_W = \{0\} \tag{2.9-18}$$

which is already in the form necessary to transform to modal coordinates.

All other elements are handled in this same manner, and need not be written explicitly. The transformation to modal coordinates is then quite simple; substitutions of the form in equations 2.9-14 are made directly into the panel point equations. It is the definition of mode shapes for the main body which might cause some debate; the choice is arbitrary, but it is felt that the above definition is the best approximation when only a few modes are retained.

In the case of a gear attached to a wing, the wing modes are not defined by 2.9-18. Panel point displacement of the gear is constrained to motion due to wing motion, constraints are written defining this gear motion, and a term such as $[T_{WG}]'[A_G][T_{WG}]$ will be added to $[A_W]$ in the final equation defining wing mode shapes.

The transformation to modal coordinates for a problem such as flutter analysis is very useful in that the modal coordinates are not coupled together in the final equations, and the simultaneous equations are easily solved on a digital computer. Here the problem is not so simple, as the rigid body motion will be coupled into each equation, and the modal coordinates are coupled in each equation. This complexity is unavoidable.

The transformed set of equations are considered next. The total number of modal coordinates which have been defined are not generally used due to the resulting large number of variables involved. It is desirable to truncate the square matrix $[\phi]$ and delete the column vector $\{q\}$ to the desired number of variables for each component. Then the transformation

$$\{P\}_i = [\bar{\phi}]_i \{\bar{q}\}_i \qquad (2.9\text{-}19)$$

is made, where $[\bar{\phi}]_i$ is the matrix formed by deleting the higher mode shape columns, and is not square. This reduction in number of variables may be treated in one of two ways. The set of equations 2.9-19 may be considered as constraint equations on the system. The right-hand side of the panel point equations would then contain total panel point forces minus the panel point forces which constrain the higher modes of deformation. This approach may be used if desired, and the associated Lagrange multipliers may be eliminated rather easily. A somewhat simpler approach may be used. Direct substitution of equation 2.9-19 into the panel point equations results in obtaining more equations than unknowns, and the solutions are not unique. However, a linear combination of these equations may be chosen which yields the proper result. The shorthand notation which defines this operation is the premultiplication of the set of equations by the transform of the deleted modal matrix, $[\bar{\phi}]_i$. The right-hand side of the equations is recognized as the contribution of the panel point forces to the modal coordinates retained, which is exactly the result obtained by the first approach.

The expansion of the total panel point displacement for the i -th component is

$$\{P\}_i = [T_{ij}]\{P\}_j + \{P^e\}_i + \{\Delta\}_i \qquad (2.9\text{-}20)$$

where it is assumed that the i -th component is attached to the vehicle by means of attaching it to the j -th component. As it is possible that the j -th component is not the fuselage, but is attached to the fuselage, then

$$\{P\}_j = [T_{jF}]\{P\}_F + \{P^e\}_j + \{\Delta\}_j \qquad (2.9\text{-}21)$$

An example is that of a gear on a wing, i corresponding to the gear and j to the wing. Using

$$\{P^e\}_i = [\bar{\Phi}]_i \{\bar{q}\}_i \tag{2.9-22}$$

$$\{P^e\}_j = [\bar{\phi}]_j \{\bar{q}\}_j \tag{2.9-23}$$

expression 2.9-18 becomes

$$\begin{aligned}\{P\}_i = {} & [T_{ij}][T_{jF}][\bar{\phi}_F]\{\bar{q}\}_F \\ & + [T_{ij}][\bar{\phi}]_j\{\bar{q}\}_j + [T_{ij}]\{\Delta\}_j \\ & + [\bar{\phi}]_i\{\bar{q}\}_i + \{\Delta\}_i\end{aligned} \tag{2.9-24}$$

One may well appreciate the complexity of writing the equations of motion, 2.8-3, 4, 5, in this notation for a general body whose geometry is as yet undefined. Hence the general set of equations using modal coordinates will not be written; rather, the transformation will be performed for individual problems as necessary to reduce the number of degrees of freedom to a level consistent with problem requirements.

2.10 POSITION AND VELOCITY OF A BODY POINT

If the time at which an applied force is imposed on the body is to be accurately known, as well as its magnitude, which may depend on the body kinematics, it is necessary that the position of the point to which it is to be applied is also known. The purpose of this section is to derive that position in terms of the variables defining the motion of the body. This may be accomplished by the integration in time of the components of the velocity of the point relative to the ground.

Let the point be labeled B. Its position in the undeflected body relative to the origin of the body coordinate system is $\mathbb{L}_B$, and its total displacement relative to the body coordinate system is $\mathbb{P}_B$. The velocity of this point may be expressed by either of the forms

$$\begin{aligned}\mathbb{V}_B &= V_{xB}\, \mathbb{i} + V_{yB}\, \mathbb{J} + V_{zB}\, \mathbb{K} \\ &= \dot{X}_B\, \mathbb{I} + \dot{Y}_B\, \mathbb{J} + \dot{Z}_B\, \mathbb{K}\end{aligned} \tag{2.10-1}$$

It may also be written as

$$\mathbb{V}_B = \frac{d\mathbb{r}_B}{dt} = \dot{\mathbb{R}} + \boldsymbol{\Omega}\times\mathbb{L}_B + \dot{\mathbb{P}}_B + \boldsymbol{\Omega}\times\mathbb{P}_B \qquad (2.10\text{-}2)$$

from equation 2.6-7. The components of the velocity of the point, expressed in the body coordinate system, are then

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}_B = \begin{Bmatrix} \mathbb{i} \\ \mathbb{j} \\ \mathbb{K} \end{Bmatrix} \cdot \frac{d\mathbb{r}_B}{dt} \qquad (2.10\text{-}3)$$

This form may be expanded into components, with the resulting expression

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}_B = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + \begin{Bmatrix} \dot{P}_x \\ \dot{P}_y \\ \dot{P}_z \end{Bmatrix}_B - [\Omega] \begin{Bmatrix} P_x + x' \\ P_y + y' \\ P_z + z' \end{Bmatrix}_B \qquad (2.10\text{-}4)$$

As the point B may be in a component other than the fuselage, the term, may be written in the component coordinate system. Then

$$\begin{Bmatrix} V_x \\ V_y \\ V_z \end{Bmatrix}_B = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + [\gamma]_B \begin{Bmatrix} \dot{P}_{x'} \\ \dot{P}_{y'} \\ \dot{P}_{z'} \end{Bmatrix}_B - [\Omega][\gamma]_B \begin{Bmatrix} P_{x'} + x' \\ P_{y'} + y' \\ P_{z'} + z' \end{Bmatrix}_B \qquad (2.10\text{-}5)$$

The components of the velocity of the point in the ground reference system are obtained by the transformation of this expression,

$$\begin{Bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{Bmatrix}_B = [\Gamma]' \left(\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + [\gamma]_B \begin{Bmatrix} \dot{P}_{x'} \\ \dot{P}_{y'} \\ \dot{P}_{z'} \end{Bmatrix}_B - [\Omega][\gamma]_B \begin{Bmatrix} P_{x'} + x' \\ P_{y'} + y' \\ P_{z'} + z' \end{Bmatrix}_B \right) \qquad (2.10\text{-}6)$$

The position of the point in the ground reference system is immediately obtained as

$$X_B(t) = X_B(0) + \int_0^t \dot{X}_B \, dt$$

$$Y_B(t) = Y_B(0) + \int_0^t \dot{Y}_B \, dt$$

$$Z_B(t) = Z_B(0) + \int_0^t \dot{Z}_B \, dt \tag{2.10-7}$$

where $X_B(0), Y_B(0), Z_B(0)$ are the coordinates of the point at zero time, the beginning of the problem. They may be found from

$$\begin{Bmatrix} X(0) \\ Y(0) \\ Z(0) \end{Bmatrix}_B = \begin{Bmatrix} X(0) \\ Y(0) \\ Z(0) \end{Bmatrix} + [\Gamma_0]'[\gamma]_B \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}_B \tag{2.10-8}$$

where $X(0), Y(0), Z(0)$ are the initial components of the position vector to the body coordinate system origin, and the zero subscript on the Eulerian transformation indicates the value of the Euler angles at zero time.

SECTION 3

APPLIED FORCES

3.1 GENERAL

A solution of the equations of motion written in Section 2.8 may be obtained when the applied forces have been defined. As the number of equations is the same as the number of variables needed to define the motion, the forces must be completely specified. They may be stated as known constants, as explicit functions of time, or as functional relations between the variables defining the motion.

The total force applied to the vehicle is defined by the vector (Q_x, Q_ψ, Q_γ). This force is the summation of all the externally applied forces on the vehicle, expressed in the body coordinate system. The contributions to this force considered in Section 3.2 are

Gravitational Force	- Q_W
Thrust Force	- Q_T
Parachute Force	- Q_P
Aerodynamic Force	- Q_A
Ground Force	- Q_G

The ground forces will be considered for a variety of contacting elements on the vehicle. The contribution to body forces is developed for a single element of each type, and the summation over the elements indicated. The total external moment on the body is defined by the vector (N_x, N_ψ, N_γ). The contribution from each of the above forces is defined, and the total moment obtained by a summation.

The definition of the vectors $(\{Q_x'\}, \{Q_\psi'\}, \{Q_\gamma'\})$ is dependent both on the mathematical model to be used, and to some extent the manner in which the problem is to be attacked. The panel point forces in each set of panel point equations must be arranged in the same order as are the panel point displacements. They may include the distributed forces which are considered to act on the whole vehicle, the internal reactions which hold the vehicle together, the damping forces which dissipate energy stored in elastic deformations, and the stroking forces between components which move as rigid bodies relative to one another. In Section 3.3, the forces on the component rigid body motions are defined in terms of ground forces and stroking forces. In Section 3.4, the forces on the panel points contributing to elastic deformations are considered, and the distinctions made as to which of these are applied forces and which are included as constraints or are included in the definition of the stiffness matrix. The discussion of constraints in Section 3.5 completes the definition of forces in the system of equations.

The contents of this section are intended to include the most common forms of applied forces which will occur in landing impact problems. Many of them will hold only for a particular configuration and are not meant to be general definitions to be applied for any problem. The individual may replace these forms with others more suited to a particular application, keeping in

mind that the forces must be completely specified either in terms of the variables already defined or as explicit functions of time.

3.2 TOTAL VEHICLE APPLIED FORCES

In this section, the forces (Q_x, Q_y, Q_z) which enter the rigid body Equations 2.8-3 will be defined. The general form for the total body force is

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix} \cdot \mathbb{Q}$$

where $\mathbb{Q}$ is the total force, given by

$$\mathbb{Q} = \mathbb{Q}_W + \mathbb{Q}_T + \mathbb{Q}_G + \mathbb{Q}_P + \mathbb{Q}_A$$

for gravitational, thrust, ground, parachute, and aerodynamic forces, respectively.

The contributions to (N_x, N_y, N_z) entering the Equations 2.8-4 will be calculated in each section; the forms vary considerably depending on the type of force. The contributions from each section may then be summed to produce the total moment.

3.2.1 Gravitational Force, $\mathbb{Q}_W$

The gravitational force acting on the vehicle is directed along the positive Z axis; the magnitude is the product of the total vehicle mass and the local acceleration due to gravity (g):

$$\mathbb{Q}_W = Mg\,\mathbb{K} \qquad (3.2\text{-}1)$$

The components of the weight vector in the body coordinate system are given in terms of the Eulerian transformation as

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_W = [\Gamma] \begin{Bmatrix} 0 \\ 0 \\ Mg \end{Bmatrix} = Mg \begin{Bmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta\cos\phi \end{Bmatrix} \qquad (3.2\text{-}2)$$

Since the origin of the body coordinate system will not be coincident with the instantaneous position of the center of mass, a moment about the body coordinate axes will result from the gravitational force. Let $\mathbb{P}_w$ be the distance from the origin of the body coordinate system to the center of mass of the total vehicle. The moment is then given by

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix}_W = \begin{Bmatrix} \mathbb{P}_w \times \mathbb{Q}_w \cdot \mathbb{I} \\ \mathbb{P}_w \times \mathbb{Q}_w \cdot \mathbb{J} \\ \mathbb{P}_w \times \mathbb{Q}_w \cdot \mathbb{K} \end{Bmatrix} = \begin{bmatrix} 0 & Q_{zw} & -Q_{yw} \\ -Q_{zw} & 0 & Q_{xw} \\ Q_{yw} & -Q_{xw} & 0 \end{bmatrix} \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}_W \qquad (3.2\text{-}3)$$

The vector $\mathbb{P}_w$ is determined in terms of the displacement from initial position of all the vehicle relative to the body coordinate system;

$$\mathbb{P}_w = \frac{\int_v \mathbb{P} \rho \, dv}{\int_v \rho \, dv} = \frac{1}{M} \int_v \mathbb{P} \rho \, dv \qquad (3.2\text{-}4)$$

The integral may be broken into integrals over each of the N components of the vehicle;

$$\int_v \mathbb{P} \rho \, dv = \sum_{i=1}^{i=n} \int_{v_i} \mathbb{P}_i \, \rho \, dv \qquad (3.2\text{-}5)$$

In terms of the matrix notation which arises from the panel point concept and the interpolation procedure, this form is rewritten. The components of the vector $\mathbb{P}_w$ are then

$$\begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}_W = \frac{1}{M} \sum_{i=1}^{i=N} \left([\gamma] \begin{bmatrix} \{1\}' [A_{x'}] \{P_{x'}\} \\ \{1\}' [A_{y'}] \{P_{y'}\} \\ \{1\}' [A_{z'}] \{P_{z'}\} \end{bmatrix} \right)_i \qquad (3.2\text{-}6)$$

where $[\gamma]$ is the transformation between the body coordinate system and the i-th component coordinate system. The moment is then obtained by substituting Equation 3.2-6 into Equation 3.2-3.

The moment calculated here is small compared to moments from the ground forces and will normally be neglected in landing impact problems.

3.2.2 Thrust Force, $\mathbb{Q}_T$

It is assumed in the discussion which follows that the magnitude of the thrust force is known explicitly as a function of time, or is a known constant, or that it is known as a function of the variables defining the vehicle motion. The line of action of the thrust vector is defined by the positions of two points on the line, labeled (a, b). The positions of these points are defined by $(\mathbb{L}+\mathbb{P})_a$ and $(\mathbb{L}+\mathbb{P})_b$ as in Fig. 3. The thrust vector then is collinear with the vector difference of these two quantities, $\mathbb{L}_T$,

$$\mathbb{L}_T = (\mathbb{L}+\mathbb{P})_a - (\mathbb{L}+\mathbb{P})_b \tag{3.2-7}$$

which may be expanded to

$$\begin{aligned} \mathbb{L}_T = {} & (x_a - x_b)\mathbb{i} + (y_a - y_b)\mathbb{J} + (z_a - z_b)\mathbb{K} \\ & + (P_{xa} - P_{xb})\mathbb{i} + (P_{ya} - P_{yb})\mathbb{J} + (P_{za} - P_{zb})\mathbb{K} \end{aligned} \tag{3.2-8}$$

The direction cosines of the vector $\mathbb{L}_T$ in the body coordinate system are then variables, given by

$$\begin{Bmatrix} \gamma_{xT} \\ \gamma_{yT} \\ \gamma_{zT} \end{Bmatrix} = \frac{1}{|\mathbb{L}_T|} \begin{Bmatrix} (x_a - x_b) + (P_{xa} - P_{xb}) \\ (y_a - y_b) + (P_{ya} - P_{yb}) \\ (z_a - z_b) + (P_{za} - P_{zb}) \end{Bmatrix} \tag{3.2-9}$$

The components of the thrust in the body coordinate system are then

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_T = Q_T \begin{Bmatrix} \gamma_{xT} \\ \gamma_{yT} \\ \gamma_{zT} \end{Bmatrix} \tag{3.2-10}$$

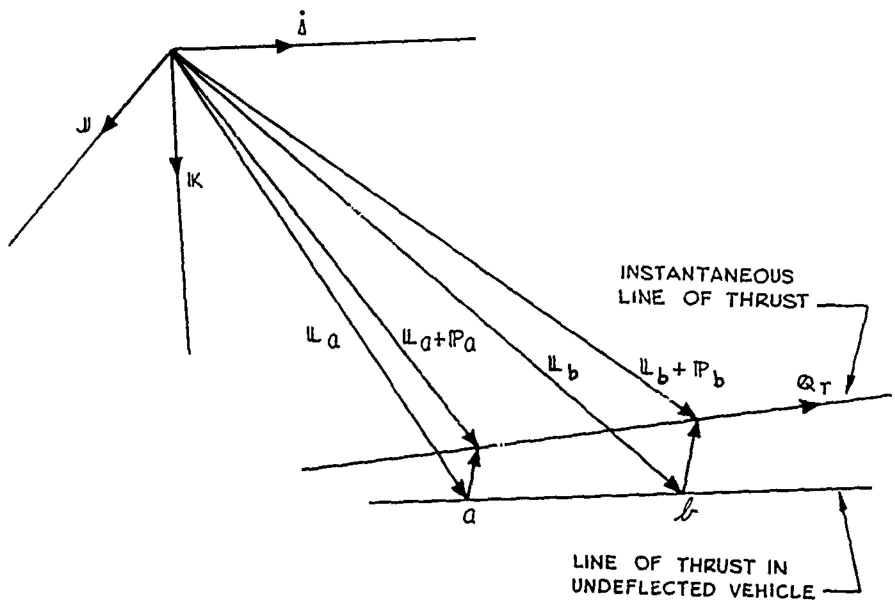


Figure 3. Line of Action of the Thrust Vector

where Q_T is the thrust magnitude. If there are several independent thrust sources, a sum of terms like this one will give the total force; if the thrust magnitudes are the same,

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_T = Q_T \sum_{K=1}^{K=n} \begin{Bmatrix} \gamma_{xT} \\ \gamma_{yT} \\ \gamma_{zT} \end{Bmatrix}_K \tag{3.2-11}$$

where K runs over the various thrust sources. The moment due to any one of these is

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix}_{T_K} = \begin{Bmatrix} (\mathbb{L}+\mathbb{P})_a \times \mathbb{Q}_T \cdot \mathbb{i} \\ (\mathbb{L}+\mathbb{P})_a \times \mathbb{Q}_T \cdot \mathbb{J} \\ (\mathbb{L}+\mathbb{P})_a \times \mathbb{Q}_T \cdot \mathbb{K} \end{Bmatrix} \tag{3.2-12}$$

$$= \begin{bmatrix} 0 & -(z_a+P_{za}) & (y_a+P_{ya}) \\ (z_a+P_{za}) & 0 & -(x_a+P_{xa}) \\ -(y_a+P_{ya}) & (x_a+P_{xa}) & 0 \end{bmatrix}_K \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_K$$

2.2.3 Parachute Force, $\mathbb{Q}_P$

Parachutes are used in conventional aircraft to shorten the ground run. For vertical alightment vehicles, the parachute may be the primary lift device, or it may only be used to ensure that vehicle orientation is essentially vertical. The general expression for the parachute force is

$$\mathbb{Q}_P = -\frac{1}{2}\rho S v_P^2 C_D \frac{\mathbb{V}_P}{|\mathbb{V}_P|} = -\frac{1}{2}\rho S C_D |\mathbb{V}_P| \mathbb{V}_P \tag{3.2-13}$$

where the various parameters are

ρ atmospheric density
$C_D S$ effective drag area of the parachute
v_P velocity of the parachute attachment point with respect to the atmosphere

In Section 2.10, the velocity of a point in the body is derived in detail. If the components of the wind velocity expressed in the body coordinate system are given as $(v_{x_w}, v_{y_w}, v_{z_w})$, then from Eq. 2.10-5, the velocity of the parachute attachment point relative to the atmosphere expressed in component form in the body coordinate system is

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}_P = -\begin{Bmatrix} v_{xw} \\ v_{yw} \\ v_{zw} \end{Bmatrix} + \left(\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + [\gamma]_P \begin{Bmatrix} \dot{P}_{x'} \\ \dot{P}_{y'} \\ \dot{P}_{z'} \end{Bmatrix}_P - [\Omega][\gamma]_P \begin{Bmatrix} P_{x'} + x' \\ P_{y'} + y' \\ P_{z'} + z' \end{Bmatrix}_P \right) \qquad (3.2\text{-}14)$$

In terms of these relative velocity components and the general parachute force expression, the parachute force is

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_P = -\frac{1}{2}\rho S C_D \sqrt{v_{xP}^{\;2} + v_{yP}^{\;2} + v_{zP}^{\;2}} \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}_P \qquad (3.2\text{-}15)$$

where the positive sign of the radical is understood.

If the parachute is to open during the time period in which the landing is analyzed, the drag area $C_D S$ will be a function of time. Thus, if the parachute is initiated at time t_i , and it takes t_d seconds to deploy and t_o seconds to open, the drag area is

$$C_D S = \begin{cases} 0 & t < t_i + t_d \\ C_D S(t) & t_i + t_d \leq t \leq t_i + t_d + t_o \\ (C_D S)_f & t_i + t_d + t_o < t \end{cases} \tag{3.2-16}$$

where $C_D S(t)$ is the growth function of the drag area during the opening period, and $(C_D S)_f$ is its final value. For conventional parachutes, these parameters may be found in reference 6.

The moment about the axes of the body coordinate system is written in component form as

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix}_P = \begin{Bmatrix} (\mathbb{L} + \mathbb{P})_P \times \mathbb{Q}_P \cdot \mathbb{I} \\ (\mathbb{L} + \mathbb{P})_P \times \mathbb{Q}_P \cdot \mathbb{J} \\ (\mathbb{L} + \mathbb{P})_P \times \mathbb{Q}_P \cdot \mathbb{K} \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & -(z_P + P_{zP}) & (y_P + P_{yP}) \\ (z_P + P_{zP}) & 0 & -(x_P + P_{xP}) \\ -(y_P + P_{yP}) & (x_P + P_{xP}) & 0 \end{bmatrix} \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_P \tag{3.2-17}$$

3.2.4 Aerodynamic Force, $\mathbb{Q}_A$

The aerodynamic forces acting on the vehicle are those forces exerted by the surrounding atmosphere resisting the motion of the vehicle . These forces are defined here in terms of the notation used in describing conventional aircraft. The contribution to these forces due to elastic motion of vehicle components will be discussed in Section 3.4. The contribution involving rigid body dynamics and control surface deflections will be considered here.

The general forms for the components of the forces and moments are

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_A = \frac{1}{2} \rho S v^2 \begin{Bmatrix} C_x \\ C_y \\ C_z \end{Bmatrix} \tag{3.2-18}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix}_A = \frac{1}{2}\rho S v^2 \begin{Bmatrix} b\,C_l \\ c\,C_m \\ b\,C_n \end{Bmatrix} \tag{3.2-19}$$

where the indicated parameters are defined by

ρ = atmospheric density
S = wing area
c = mean aerodynamic chord
b = wing span
v = velocity of the vehicle relative to the atmosphere

$\left.\begin{matrix} C_x \\ C_y \\ C_z \end{matrix}\right\}$ = non-dimensional aerodynamic force coefficients referred to the body axis

$\left.\begin{matrix} C_l \\ C_m \\ C_n \end{matrix}\right\}$ = non-dimensional aerodynamic moment coefficients referred to the body axis

Each of the aerodynamic coefficients is a function of the body orientation and velocities, control surface deflections and velocities, and elastic body motion. For the vehicle without elastic motions,

$$C_i = C_i\left(\alpha, \beta, v_x, v_y, v_z, \Omega_x, \Omega_y, \Omega_z, \delta, \dot{\delta}\right) \tag{3.2-20}$$

where α, the angle of attack, and β, the sideslip angle, define the orientation of the free stream velocity v with respect to the body x-axis. The control surface deflection, δ, and the corresponding velocity, $\dot{\delta}$, are considered here for an arbitrary control surface. The subscript i refers to any of the above-mentioned coefficients.

In the theory of aerodynamics, it is usually adequate to assume that the aerodynamic coefficients are linear in each of the variables. This is equivalent to expanding the coefficient in a Taylor's series and omitting all non-linear terms;

$$C_i = C_i(0) + \frac{\partial C_i}{\partial \alpha}(0)\,\alpha + \frac{\partial C_i}{\partial \beta}(0)\,\beta + \cdots\cdots\cdots \tag{3.2-21}$$

The higher order terms are assumed negligible in the linear theory.

For landing response problems, not all of the linear terms are appreciable in a given coefficient. In the following list of coefficients, only the most important terms have been retained.

$$C_x = C_{x_0} + \begin{Bmatrix} \partial C_x/\partial\alpha \\ \partial C_x/\partial\delta_E \\ \partial C_x/\partial\delta_F \end{Bmatrix}_0' \begin{Bmatrix} \alpha \\ \delta_E \\ \delta_F \end{Bmatrix} \tag{3.2-22}$$

$$C_y = \begin{Bmatrix} \partial C_y/\partial\beta \\ \partial C_y/\partial\delta_R \end{Bmatrix}_0' \begin{Bmatrix} \beta \\ \delta_R \end{Bmatrix} \tag{3.2-23}$$

$$C_z = C_{z_0} + \begin{Bmatrix} \partial C_z/\partial\alpha \\ \partial C_z/\partial\delta_E \\ \partial C_z/\partial\delta_F \end{Bmatrix}_0' \begin{Bmatrix} \alpha \\ \delta_E \\ \delta_F \end{Bmatrix} \tag{3.2-24}$$

$$C_\ell = \begin{Bmatrix} \partial C_\ell/\partial\delta_A \\ \partial C_\ell/\partial\left(\frac{\Omega_x b}{2v}\right) \end{Bmatrix}_0' \begin{Bmatrix} \delta_A \\ \left(\frac{\Omega_x b}{2v}\right) \end{Bmatrix} \tag{3.2-25}$$

$$C_m = C_{m_0} + \begin{Bmatrix} \partial C_m/\partial\alpha \\ \partial C_m/\partial\delta_E \\ \partial C_m/\partial\delta_F \end{Bmatrix}_0' \begin{Bmatrix} \alpha \\ \delta_E \\ \delta_F \end{Bmatrix} \tag{3.2-26}$$

$$C_n = \begin{Bmatrix} \partial C_n / \partial \beta \\ \partial C_n / \partial \delta_R \end{Bmatrix}_0' \begin{Bmatrix} \beta \\ \delta_R \end{Bmatrix} \tag{3.2-27}$$

The indicated variables are

δ_E = pitch control surface deflection, such as elevator, ailevator
δ_F = flap deflecton
δ_R = rudder deflection
δ_A = roll control surface deflection, such as aileron or spoiler

The values $C_{x_0}, C_{z_0}, C_{m_0}$ for zero deflections and the partial derivatives evaluated at zero deflections are obtained from wind tunnel tests.

The above terms are obviously not all-inclusive, but should provide an adequate definition for most landing problems. A complete treatment of aerodynamic forces may be found in reference 7 . The control surface deflections are assumed to be known as constants or explicit functions of time.

The only other matter to be covered in this Section is the calculation of the angles of attack and sideslip. The velocity which enters Equations 3.2-18, 19, which define the force and moment on the body, is the difference between the body translational velocity and wind velocity, which yields

$$v^2 = \begin{Bmatrix} v_x - v_{xw} \\ v_y - v_{yw} \\ v_z - v_{zw} \end{Bmatrix}' \begin{Bmatrix} v_x - v_{xw} \\ v_y - v_{yw} \\ v_z - v_{zw} \end{Bmatrix} \tag{3.2-28}$$

The angle of attack, α, is defined as the angle between the velocity vector v projected into the x-z plane and the x-axis. This requires

$$\alpha = \tan^{-1} \frac{v_z - v_{zw}}{v_x - v_{xw}} \tag{3.2-29}$$

Similarly the sideslip angle is

$$\beta = \tan^{-1} \frac{v_y - v_{yW}}{v_x - v_{xW}} \tag{3.2-30}$$

For conventional aircraft landings, the forward velocity, v_x, is much larger than the drift or sinking speeds, and the following relations are very nearly true.

$$v^2 = (v_x - v_{xW})^2 \tag{3.2-31}$$

$$\alpha = \frac{v_z - v_{zW}}{v_x - v_{xW}} \tag{3.2-32}$$

$$\beta = \frac{v_y - v_{yW}}{v_x - v_{xW}} \tag{3.2-33}$$

3.2.5 Ground Forces, Q_G

In this Section, forces on the body due to interaction with the ground will be developed. The section will be divided according to the type of contacting element under consideration. These elements will be of several types: surface pods, tires, skis and skids, spikes, and gas-filled bags.

In some areas, the ground will be considered to be rigid. In these cases, the contacting element is considered to develop a coefficient of friction with respect to the ground, which may be dependent on parameters of the motion. In other areas, the ground is allowed flexibility, viscosity, or compressibility to varying extents.

The ground coordinate system is used in describing most of these forces. The components in the ground coordinate system of the force exerted on the body by the ground, called ground reactions, are $(D_G, S_G, V_G)_m$, where the subscript indicates a particular contacting element on the vehicle. The total force on the body due to the ground is then

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_G = [\Gamma] \sum_{m} \begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix}_m \tag{3.2-34}$$

The separate contributions to force on the body are

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_{G_m} = [\Gamma] \begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix}_m \tag{3.2-35}$$

Moments about the axes of the body coordinate system are found from the general form

$$\mathbb{N}_G = \sum_{m} (\mathbb{L}_m + \mathbb{P}_m) \times \mathcal{Q}_m \tag{3.2-36}$$

where $(\mathbb{L}_m + \mathbb{P}_m)$ is the instantaneous lever arm of the force $\mathcal{Q}_m$. This is written in component form as

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix}_G = \sum_{m} \begin{Bmatrix} (y+P_y)Q_{zG} & -(z+P_z)Q_{yG} \\ (z+P_z)Q_{xG} & -(x+P_x)Q_{zG} \\ (x+P_x)Q_{yG} & -(y+P_y)Q_{xG} \end{Bmatrix}_m \tag{3.2-37}$$

where the forces and lever arms for each contacting element are indicated in component form.

The forces as written in the following pages will be written for a single contacting element, with no subscript attached to identify the element. It will then be understood that these forms must be written for each contacting element when making use of the Equations 3.2-34, 37 for body forces and moments.

3.2.5.1 Surface pads

A surface pad is a mass rigidly attached to the end of the gear piston. It is not allowed a degree of freedom for motion, but simply moves with the piston. It has sufficient contacting area so that its ground penetration is small if the ground is considered to be soft.

The ground force will first be described for the case when the ground is rigid. The surface pad will develop a coefficient of friction with respect to the ground. The ground force will be developed entirely in terms of the body motion. The force from the ground along the stroking axis is first developed.

The drag force on the pad is opposite to the direction of the horizontal pad velocity, which requires

$$D_G \dot{Y}_P - S_G \dot{X}_P = 0 \tag{3.2-38}$$

where the components of the pad velocity in the ground plane are $(\dot{X}_P, \dot{Y}_P)$, the force on the pad parallel to the X-axis is D_G, and that along the Y-axis is S_G. The magnitude of the drag force is the product of vertical force on the ground with the coefficient of friction, yielding

$$D_G^{\,2} + S_G^{\,2} = \mu^2 V_G^{\,2} \tag{3.2-39}$$

These equations may be combined to yield

$$\begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix} = V_G \begin{Bmatrix} \mu \dfrac{\dot{X}_P}{|\dot{X}_P|}\left[1 + (\dot{Y}_P/\dot{X}_P)^2\right]^{-1/2} \\ \dfrac{\dot{Y}_P}{|\dot{Y}_P|}\left[1 + (\dot{X}_P/\dot{Y}_P)^2\right]^{-1/2} \\ 1 \end{Bmatrix} \tag{3.2-40}$$

where the forms in velocity components divided by their magnitudes resolve the arbitrariness in sign in solving the quadratic Equation 3.2-39. The forces on the ground are related to the forces on the pad by

$$\begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix} = [\Gamma]' [\gamma]_P \begin{Bmatrix} Q_{x'} \\ Q_{y'} \\ Q_{z'} \end{Bmatrix}_P \qquad (3.2\text{-}41)$$

which may be inverted to give

$$Q_{z'GP} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}' [\gamma]'_P [\Gamma] \begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix} \qquad (3.2\text{-}42)$$

The force $Q_{z'GP}$ is along the stroking axis of the gear. After contact, there is no motion of the piston relative to the ground along this axis. Then $Q_{z'GP}$ is the negative of the piston stroking force, $Q_{z's}$, which is positive downward along the stroking axis. The stroking force is the sum of the internal gear axial forces to be developed later. The Equations 3.2-40, 42 combine with the above definition to relate the vertical force on the ground to the piston stroking force:

$$V_G = \frac{-Q_{z's}}{\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}' [\gamma]' [\Gamma] \begin{Bmatrix} \mu(\dot{X}_P/|\dot{X}_P|)[1+(\dot{Y}_P/\dot{X}_P)^2]^{-1/2} \\ \mu(\dot{Y}_P/|\dot{Y}_P|)[1+(\dot{X}_P/\dot{Y}_P)^2]^{-1/2} \\ 1 \end{Bmatrix}} \qquad (3.2\text{-}43)$$

Combining Equations 3.2-40, 43 yields the ground force for a single contacting element,

$$\begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix} = \frac{-Q_{z's} \begin{Bmatrix} \mu(\dot{X}_p/|\dot{X}_p|)[1+(\dot{Y}_p/\dot{X}_p)^2]^{-1/2} \\ \mu(\dot{Y}_p/|\dot{Y}_p|)[1+(\dot{X}_p/\dot{Y}_p)^2]^{-1/2} \\ 1 \end{Bmatrix}}{\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}' [\gamma]'[\Gamma] \begin{Bmatrix} \mu(\dot{X}_p/|\dot{X}_p|)[1+(\dot{Y}_p/\dot{X}_p)^2]^{-1/2} \\ \mu(\dot{Y}_p/|\dot{Y}_p|)[1+(\dot{X}_p/\dot{Y}_p)^2]^{-1/2} \\ 1 \end{Bmatrix}} \qquad (3.2\text{-}44)$$

From this expression and the Equations 3.2-34, 37 the body forces and moments may readily be calculated.

This form for ground forces is quite complicated; it may, however, be simplified for most applications. One of the examples in Section 4 will make use of this form, and the simplifications used there will perhaps enlighten the reader.

If the ground is soft, the above statements will not be applicable. It is suggested that the following forms for ground force be used. The vertical ground force may be represented by a one-way spring, that is,

$$V_G = \begin{cases} -k_1 \, d & \dot{Z}_p > 0 \\ 0 & Z_p, \dot{Z}_p \leq 0 \end{cases} \qquad (3.2\text{-}45)$$

where the velocity of the surface pad normal to the ground is obtained from Equation 2.10-6. The spring rate k_1 must be obtained from experiment, as it will depend on the geometry of the pad and characteristics of the ground. The penetration depth d is obtained by integration of $\dot{Z}_p$ following touchdown.

The ground forces in the ground plane may be represented by

$$D_G = \begin{cases} -k_2 \, \dot{X}_p \, d & d > 0 \\ 0 & d \leq 0 \end{cases} \qquad (3.2\text{-}46)$$

$$S_G = \begin{cases} -k_2 \dot{Y}_p d & d > 0 \\ 0 & d \leq 0 \end{cases} \qquad (3.2\text{-}47)$$

where the velocities are obtained from Equation 2.10-6 and the constant k_2 must be determined from experiment. The experimental problems might be avoided by use of some analytical formulas in terms of pad area and ground characteristics; a form of this sort may be useful in prediction of peak loads, but must be used with caution in investigating stability.

3.2.5.2 Tires

Forces generated in the system due to interaction of a tire and the ground have never been very accurately described on a theoretical basis. Generally, experimental data furnished by the manufacturer is more reliable than purely theoretical data, and such should be used when available. These data may include vertical load-deflection characteristics, coefficient of sliding friction, cornering coefficient, self-aligning torque coefficient, or other. The forms presented here are in general taken from the available literature and are felt to be consistent with the state of the art concerning tire characteristics.

The vertical load-deflection characteristics of a tire will be discussed first. The forces in the ground plane will then be discussed for the periods in time before and after wheel spinup.

The ground is assumed to be rigid throughout this development.

Vertical Ground Force, V_G

The force on the body normal to the ground plane is written here in terms of physical parameters of the tire, and should be used only if data from an experimental program are not available. The variation in ground force with tire deflection has been written in many ways, each formula representing an approximate fit to the observed data. Hadekel (Ref. 8) notes that the most rational approach to vertical load-deflection characteristics produces the form

$$V_G = \begin{cases} 0 & \delta < 0 \\ -\overline{A}\,(\overline{P} + \overline{P_c} + \overline{\Delta P}) & 0 < \delta < \delta_b \end{cases} \qquad (3.2\text{-}48)$$

where the tire contacting area, $\overline{A}$, is given by

$$\overline{A} = 2.25(\delta - .03\,w)\sqrt{\overline{D}\,w} \qquad (3.2\text{-}49)$$

and the various other quantities are

δ = tire vertical deflection
w = tire width
$\overline{P}$ = undeflected tire pressure
$\overline{P_c}$ = tire wall equivalent pressure
$\overline{\Delta P}$ = pressure rise on deflection
$\overline{D}$ = tire outer diameter
δ_b = tire deflection on initial bottoming

The change in pressure with deflection is found in the static case to be

$$\overline{\Delta P} = \overline{P} \quad u\left(\frac{\delta}{w}\right)^2 \qquad (3.2\text{-}50)$$

where

$$u = \frac{\sqrt{\frac{w}{\overline{D}}}}{2.2\left(1 - \frac{w}{D}\right)} \qquad (3.2\text{-}51)$$

This approach yields results negligibly different from dynamic loads obtained from drop tests for low deflections; the dynamic loads rise more rapidly for large deflections. This may be corrected to some extent by using a polytropic compression form such as an adiabatic compression would yield. The above forms are inadequate for tire bottoming in any case. It is then necessary to replace Equation 3.2-48 by the form

$$V_G = V_{G_b} - k_b(\delta - \delta_b) \qquad (3.2\text{-}52)$$

$$= (V_{G_b} + k_b\,\delta_b) - k_b\,\delta \qquad \delta > \delta_b$$

where V_{Gb} is the vertical ground force wh . the tire first bottoms and k_b is the tire bottoming spring rate.

There are several points which should be considered here. Hysteresis effects, which may be observed in standing tires, are considered by Hadekel to be negligible in dynamics problems with high spinup speeds; hence equivalent damping in tires will not be considered. Tire lateral deflections produce vertical deflections for constant vertical load, and so both should be specified in order to find vertical load. The omission of this effect will not appreciably change the time history of the load, except that the time for tire bottoming may be in slight error.

It has been assumed here that the vertical tire deflection in the wheel plane and the "average" deflection observed in a tire with the wheel plane slightly tilted from the vertical are identical. Thus, the load deflection characteristics are assumed independent of wheel plane orientation for small angles from the vertical.

The vertical deflection is found as follows. The distance to the axle from the ground, which will be negative according to the sign convention, is

$$\mathbb{Z}_A = \left(\mathbb{R} + \mathbb{L}_A + \mathbb{P}_A\right) \cdot \mathbb{K} \tag{3.2-53}$$

where $\mathbb{L}_A, \mathbb{P}_A$ are the undeflected position and the deflection of the axle. This may be written in component form as

$$\mathbb{Z}_A = \mathbb{Z} + \begin{Bmatrix} x' + P_{x'} \\ y' + P_{y'} \\ z' + P_{z'} \end{Bmatrix}_A' [\gamma]_A' [\Gamma] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{3.2-54}$$

where $\mathbb{Z}$ is the position of the origin of the body coordinate system, and the subscript refers to the axle and the coordinate system of the gear to which it belongs. The instantaneous angle which the axle makes with the ground, α_A defines the projection of the undeflected tire radius on the normal to the ground, which yields the deflection as

$$\delta = \mathbb{Z} + \begin{Bmatrix} x' + P_{x'} \\ y' + P_{y'} \\ z' + P_{z'} \end{Bmatrix}_A' [\gamma]' [\Gamma] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \bar{r} \cos \alpha_A \tag{3.2-55}$$

For normal aircraft landings, the angle α_A is small, so that $\cos\alpha_A$ is very nearly unity. The error here is of the same order as that in using the deflection obtained in the formula presented by Hadekel.

For most aircraft landings, the Euler angles remain small, and the Eulerian transformation may be linearized. The resulting expression for tire deflection is then

$$\delta = Z + \begin{Bmatrix} x' + P_{x'} \\ y' + P_{y'} \\ z' + P_{z'} \end{Bmatrix}_A' [\gamma]' \begin{Bmatrix} -\theta \\ \phi \\ 1 \end{Bmatrix} + \bar{r} \tag{3.2-56}$$

The vertical ground force as written here is valid both before and after wheel spinup. The forces on the ground in the ground plane, however, are dependent on whether or not the wheel is spun up. In these areas, the load-deflection characteristics lateral and tangential to the tire will not be used in the sense of Equation 3.2-48. Pre-spinup forces are derived from the sliding coefficient of friction, the vertical force, and the geometry. Post-spinup force along the wheel plane is considered negligible, and that normal to the wheel plane is found from the cornering characteristics of the tire.

<u>Ground Forces in the Ground Plane,</u> D_G. S_G

Forces from the ground along the X -axis and Y -axis, D_G and S_G, are usually defined in two separate regions: pre-spinup and post-spinup. In the region of transition between the two, a form may be picked which is intuitively satisfying in order to obtain continuous forcing functions. A search of the literature shows that no analytical procedures derivable from physical laws are in existence for this region, and no experimental data is of sufficient accuracy to define any variation with tire parameters.

It is also convenient to present in this section the moment about the axle during spinup.

<u>Pre-Spinup Forces in the Ground Plane</u>

The forms derived here are based on the assumption that the force from the ground, in the ground plane, is in the opposite direction of the axle velocity vector parallel to the ground, $\mathbb{V}_A$, given by

$$\mathbb{V}_A = \dot{X}_A \, \mathbb{I} + \dot{Y}_A \, \mathbb{J} \tag{3.2-57}$$

The magnitude of the force is assumed to be the product of vertical force with the sliding coefficient of friction between the tire and the ground. The

components of the force, as pictured in Fig. 4, are then

$$D_G = \mu_S V_G \frac{\dot{X}_A}{|V_A|} \qquad t < t_t \tag{3.2-58}$$

$$S_G = \mu_S V_G \frac{\dot{Y}_A}{|V_A|} \qquad t < t_t \tag{3.2-59}$$

where t_t is the spinup time. The components of the axle velocity may be found from Equation 2.10-6. For conventional aircraft landings, the forward velocity is larger than any other, and the approximation $|V_A| = |\dot{X}_A|$ is valid. Otherwise, the root-sum-square representing the velocity magnitude must be used.

Some discussion of the variation of the sliding coefficient of friction with velocity, pressure, and temperature is worthwhile here. Hample (Ref. 9) presents experimental data for pressure, load, and temperature variations of the static coefficient of friction for tire materials. Velocity variations are presented by Luthman (Ref.10) and Gough (Ref. 11), et al. The data by Hample show that the static coefficient decreases sharply from room temperature to 300° F, then gradually to 500° F, and again sharply to the melting point. Material was taken from a B-29 nose wheel tire. He also notes that the static coefficient decreases with increasing normal pressure, the decrease becoming sharper with increasing temperature. This variation is substantiated by Luthman, although his main interest is the variation with velocity. This is characterized by a general decrease as velocity increases, but with oscillations superimposed. Luthman states that this phenomenon has been noted previously, but that no explanation has been found. Gough, et al., present similar although less extensive data points; the oscillations seem less dominant, and they present a smooth curve through the data points.

These statements describe the variations in the coefficient of friction in a general manner. Theoretical work on the subject is generally avoided due to the many coupled variables involved. Since the relative velocity of the tire footprint and the ground varies from air speed to zero during spinup, it is advisable to include the variation of the coefficient with velocity. A linear term in contact pressure may also be included. Denoting the relative velocity by v_R and the contact pressure by P_{CT}, the variation may be approximated by

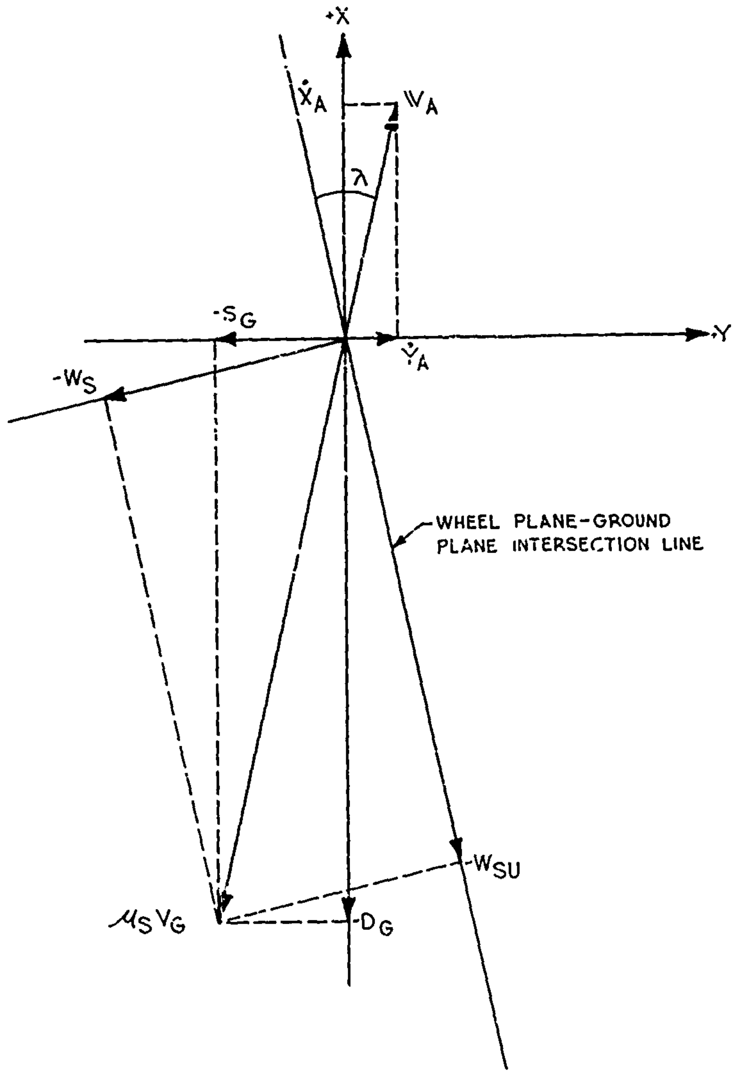


Figure 4. Tire Forces in the Ground Plane

$$\mu_S(v_R \cdot P_{CT}) = B(1 - b\,P_{CT})\left(1 + \frac{C-1}{1 + a v_R}\right) \qquad (3.2\text{-}60)$$

where B is the value for low pressure and high velocity, C is the ratio of values at high to low velocities, and a, b must be determined from experiment. Contact pressure as a function of vertical deflection is derived from Equation 3.2-48, the ratio of vertical load to footprint area. To the extent to which the variation in coefficient of friction with velocity is known, the relative velocity may be approximated by

$$v_R = |V_A| - \bar{r}\,\dot{n} \qquad (3.2\text{-}61)$$

It will be noted later that these forms may be too complicated for ordinary use, but simplified forms can be used.

The above equations are all valid for $t < t_t$, where t_t, the spinup time, is found from solution of the equations of motion. If desired, spinup time may be approximated with use of a formula by Flugge (Ref. 12):

$$t_t = \frac{1}{\bar{r}}\sqrt{\frac{2 I v_x}{\mu_S \dot{F}} - \frac{2 I}{k}} \qquad (3.2\text{-}62)$$

where I is the wheel moment of inertia, k is the gear forward spring rate, and $\dot{F}$ is the vertical load time derivative (assumed constant).

It is useful to insert at this point the forces along and normal to the line of intersection of the wheel plane and the ground plane, although they are not used in determining body forces.

The force along the line is the force used in defining the spinup moment, and is designated W_{SU}. The force at right angles is the side force, designated W_S. These are forces on the tire, given by

$$W_{SU} = \mu_S V_G \cos\lambda \qquad t < t_t \qquad (3.2\text{-}63)$$

$$W_S = \mu_S V_G \sin\lambda \qquad t < t_t \qquad (3.2\text{-}64)$$

where the angle λ , shown in Fig. 4, is given by

$$\lambda = -\psi - \beta + \sin^{-1}\left(\frac{\dot{Y}_A}{\mathbb{V}_A}\right) \tag{3.2-65}$$

The angle β is that through which the axle is rotated about the gear centerline due to torsional elasticity, and ψ is defined in the Eulerian transformation. The spinup moment is given by

$$N_{su} = (\bar{r} - \delta) W_{su} \tag{3.2-66}$$

Note that for conventional airplane landings, $\dot{Y} \ll \dot{X}$, hence

$$\sin^{-1} \frac{\dot{Y}_A}{|\mathbb{V}_A|} \simeq \frac{\dot{Y}_A}{\dot{X}_A} \tag{3.2-67}$$

Post-Spinup Forces in the Ground Plane

The force from the ground in the ground plane following spinup is usually considered to be normal to the line of intersection of ground plane and wheel plane. This assumes rolling friction is negligible and wheel braking is not present. The side force for conventional aircraft landings is

$$W_s = -K\lambda \qquad\qquad t > t_t \tag{3.2-68}$$

where K is the tire cornering coefficient and λ is the slip angle, defined in Equations 3.2-65, 67. If the slip angle and vertical force are small, this form is valid with the cornering coefficient constant. For high vertical loads, a method (Ref. 13) has been devised to yield a cornering coefficient dependent on tire vertical deflection,

$$K = \frac{\pi \bar{a} P(\omega + G)}{G} \left[\left(1.11 - \frac{1.91\,\delta}{G}\right)\omega + G\right] \tag{3.2-69}$$

where

$$\omega = \sqrt{2\bar{r}\,\delta - \delta^2} \tag{3.2-70}$$

The remaining various parameters are

$\bar{a}$ = tire section radius
P = inflation pressure
d = bead seat radius
$\bar{r}$ = tire undeflected radius
$G = \bar{r} - d$
δ = tire vertical deflection

This approach is valid for high vertical loads prior to tire bottoming, but will still be restricted to small slip angles. As the slip angle increases, the side load reaches a maximum, then decreases to the value which would occur if the tire were skidding laterally. Hadekel presents some data on this effect; no theoretical forms for high slip angles have been developed.

Forces along the ground coordinate axes for the case where $\lambda \ll 1$ are then given by

$$D_G = W_S \left(\lambda - \dot{Y}_A / \dot{X}_A\right) \qquad t > t_t \tag{3.2-71}$$

$$S_G = W_S \qquad t > t_t \tag{3.2-72}$$

Ref. 12 also gives the tire self-aligning torque M arising from slip angle λ as

$$M = m\,\lambda \tag{3.2-73}$$

in the range $\lambda < 5^\circ$, where

$$m = 12\,\bar{a}\,(1.57\,G - \bar{a})(\omega + G)P \tag{3.2-74}$$

3.2.5.3 Spike-Soil forces

Penetration on impact is a problem not easily handled, and is not discussed in the literature which is readily available. However, an approximate form may be developed which is useful for impact on sand or hard soils.

Penetration forces may be broken into two groups: compression forces and friction forces. Compression forces are described as follows.

The bulk modulus of a medium is defined as the change in pressure per unit volume on compression of the medium:

$$B = V \frac{dP}{dV} \tag{3.2-75}$$

This definition is valid only for static pressures and confined volumes. Ref. 14 notes that dynamic values of the bulk modulus of sandy soils are 200 to 300 percent of the static value. The static form will be used to formulate the compression forces on impacting; the volume, volume change, and pressure change will be considered time dependent, and the value of the dynamic bulk modulus will be used for calculations.

Consider a body with velocity v and cross-sectional area A normal to the velocity vector. As this area moves through the medium an amount ds, a compression wave travels outward radially from the area. The volume encompassed by the wave is

$$V = \frac{2}{3}\pi \delta^3 \tag{3.2-76}$$

where δ is the distance from the area to the wave front. The pressure change due to expansion of the wave front an amount $d\delta$ is

$$dP = F d\left(\frac{1}{A}\right) = F d\left(\frac{1}{2\pi\delta^2}\right) = -F \frac{d\delta}{\pi\delta^3} \tag{3.2-77}$$

so that

$$V dP = -\frac{2}{3} F d\delta \tag{3.2-78}$$

This form is independent of the radius of the compression wave. The radius may then be considered small, and the force F as that force on the soil produced by the motion of the area A . The change in volume of the soil is related to the cross-sectional area A by the penetration dS which occurs in the time for the pressure change dP to occur:

$$dV = -A\,dS \tag{3.2-79}$$

The change in pressure wave radius may be written

$$d\delta = V\,dt = \frac{V}{v}\,ds \tag{3.2-80}$$

where V is the velocity of the compression wave in the medium, since S may be considered implicitly dependent on time. The bulk modulus is then

$$B = -\frac{2}{3}\,\frac{F}{A}\,\frac{V}{v} \tag{3.2-81}$$

where the force F is now that on the area A , and not on the soil, since a minus sign has been added. The force opposing the velocity v is then

$$F = -\left(\frac{3}{2}\,\frac{B}{V}\right)A\,v \tag{3.2-82}$$

This form will be used to calculate forces and moments on a spike. In the derivation the assumption has been made that B and V are independent of penetration depth, and that no local compacting will occur. The omission of these factors will not affect the portion of a landing during which peak loads occur. However, the stability of a body landing on a single spike is determined from restraining moments on the spike alone, and this area will not be described too accurately.

Spike forces will be calculated for a single spike, symmetrically located in a symmetrical body, from the form in Equation 3.2-82. For the spike problem to be formulated, the body z-axis will remain in a plane normal to the ground,

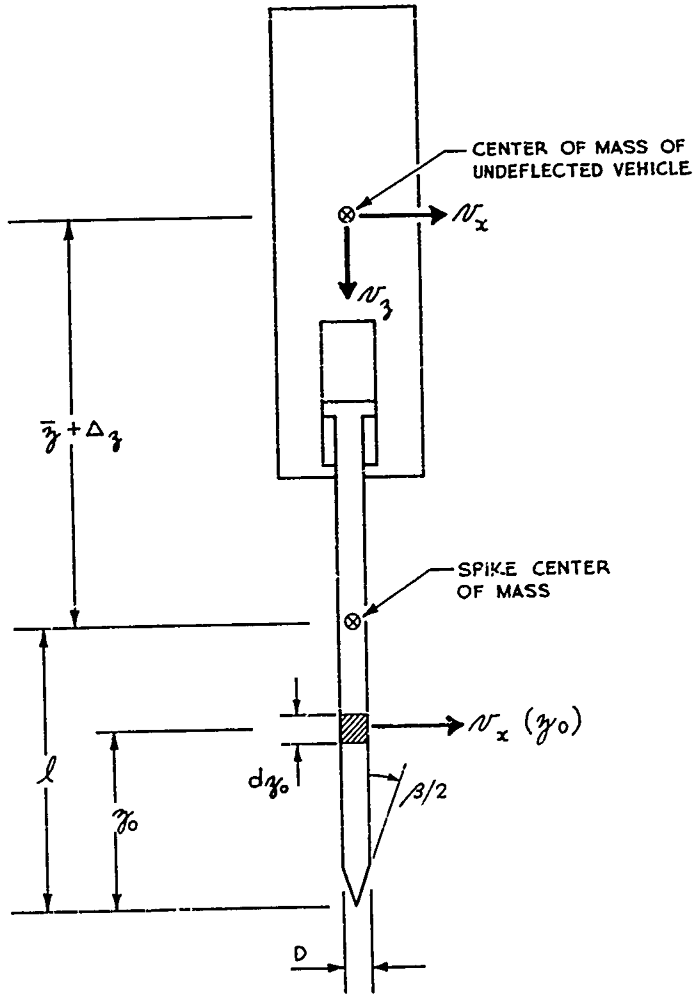


Figure 5. Spike Geometry

and the body may rotate about the y -axis only. There will then be two components of force on the body and one component of moment. The extension from this problem to one in which a vehicle is supported by several spikes will not be explicitly performed here. The spike will be considered rigid in this derivation. Consideration of spike flexibility will require more rigorous definition of the ground forces.

The geometry for the problem is depicted in Fig. 5. The spike forces and moments will be calculated directly in the body coordinate system, as they are dependent only on relative velocities.

The component of force along the spike axis, or body z-axis, is immediately written as

$$Q_{zG} = -\left(\frac{3}{2}\frac{B}{V}\right)A(d)\left(v_z + \dot{\Delta}_z\right) \tag{3.2-83}$$

where $\dot{\Delta}_z$ is the stroking velocity of the piston relative to the body, and $A(d)$ is the cross-sectional area of the spike normal to the axis, as a function of penetration depth d. The leading edge of the spike will be pointed; hence

$$A(d) = \begin{cases} \pi d^2 \tan^2(\beta/2) & d \leq \sqrt{\dfrac{A_0}{\pi}} \cot\beta/2 \\ A_0 & d \geq \sqrt{\dfrac{A_0}{\pi}} \cot\beta/2 \end{cases} \tag{3.2-84}$$

for a conical point, where A_0 is the maximum cross-sectional area, β is the apex angle, and the penetration depth is given by

$$d(t) = \int_0^t \left(v_z + \dot{\Delta}_z\right) dt \tag{3.2-85}$$

Time is considered to start on impact.

Neglecting the sharpened point, the cross-sectional area normal to the body x -axis is $dA = D\,dz_0$, where D is the spike diameter and dz_0 is an incremental distance measured along the spike, z_0 being measured from the tip. The lateral force on this area is

$$dQ_{xG}\,(z_0) = -\frac{3}{2}\,\frac{B}{V}\,D\,\;v_x\,(z_0)\,dz_0 \tag{3.2-86}$$

The lateral velocity $v_x(z_0)$ is given by

$$v_x\,(z_0) = v_x + \Omega_y\,(\bar{z} + \Delta_z + \ell - z_0) \tag{3.2-87}$$

and the total lateral force on the body is

$$\begin{aligned} Q_{xG} &= -\frac{3}{2}\,\frac{B}{V}\,D\int_0^d v_x\,(z_0)\,dz_0 \\ &= -\frac{3}{2}\,\frac{B}{V}\,D\left[\left(v_x + \Omega_y(\bar{z} + \Delta_z + \ell)\right)d - \frac{\Omega_y d^2}{2}\right] \end{aligned} \tag{3.2-88}$$

The distance $\bar{z}$ is from the origin of the body coordinate system, or center of mass of the undeflected body, to the center of mass of the piston.

The moment about the body y -axis due to dQ_{xG} is

$$dN_{yG} = (\bar{z} + \Delta_z + \ell - z_0)\,dQ_{xG} \tag{3.2-89}$$

which integrates to

$$N_{yG} = (\bar{z} + \Delta_z + \ell - d/2)\,Q_{xG} - \frac{BD}{V}\,\frac{\Omega_y d^3}{8} \tag{3.2-90}$$

The moment about the piston center of mass is

$$N_{yPG} = (\ell - d/2)\,Q_{xG} - \frac{BD}{V}\,\frac{\Omega_y d^3}{8} \tag{3.2-91}$$

3.2.5.4 <u>Skis and skids</u>

The elements referred to as skis or skids will not be distinguished here, as they have essentially the same properties. Either will be a device of some length attached to the lower end of a gear, oriented such that the rearward end contacts the ground first. The element will generally be allowed to rotate relative to the gear, and will have some sort of rotational spring damper to control this motion.

In the case of rigid ground, the ground forces will be derived from

elimination of a degree of freedom for motion of the contacting element, as in the case of the surface pads on a rigid ground. If the ground is not rigid, ground forces may be developed using the ground penetration development expressed in Paragraph 3.2.5.3.

Skis and Skids, Rigid Ground

If the ground is to be considered rigid, the ground forces cannot be derived from properties of the ground, and must be determined from the motion of the contacting element, as was done in Section 3.2.5.1 for surface pads on a rigid ground.

The rotational motion of the element about the axle as a rigid body is governed by Equation 2.8-36. However, if the moment of inertia of the element about the axle is sufficiently small, the inertial terms may all be ignored, and the restraining moment about the axle may be set equal to the moment about the axle from the ground forces. The degree of freedom of the rotation is then eliminated as a variable in order to yield the ground force. If the restraining moment is only for the purpose of opposing rebound and not to absorb landing impact, the forces involved before the element flattens on the ground are actually small at any rate. The element must, of course, be considered rigid in this case.

The geometry is depicted in Fig. 6. The force W_N is the component of the total ground force acting to rotate the element about its axle. This component is not generally in a plane normal to the ground plane, as the figure might indicate. It is first necessary to develop the form for this component. The axle is assumed to be parallel to the body y-axis initially. The element may rotate about the gear center line, or the η'-axis, through an angle β (assumed small) due to torsional elasticity. Its angle relative to the x'-axis of the gear coordinate system is η; consistent with the definition used in forming Eq. 2.8-39 governing component rigid body motion about a line. The coordinate system in which, instantaneously, the x''-axis lies parallel to the element and the y''-axis lies parallel to the axle, is derived by the product of transformations due to the rotations β, η from the gear coordinate system;

$$\begin{Bmatrix} \mathbf{i}'' \\ \mathbf{j}'' \\ \mathbf{k}'' \end{Bmatrix} = \begin{bmatrix} \cos\eta & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{bmatrix} \begin{bmatrix} 1 & \beta & 0 \\ -\beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{Bmatrix} \qquad (3.2\text{-}92)$$

The unit vectors in the gear coordinate system may be written in terms of those in the ground coordinate system by

$$\begin{Bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{Bmatrix} = [\gamma]'[\Gamma] \begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} \qquad (3.2\text{-}93)$$

The components of the ground force in the contacting element coordinate system are then related to those in the ground coordinate system by

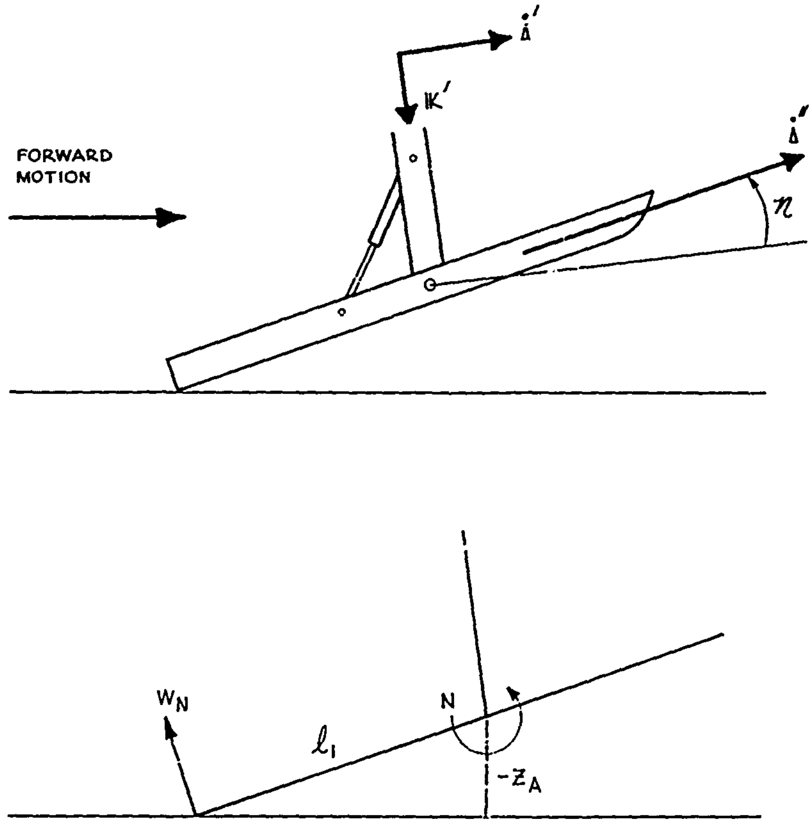


Figure 6. Ski or Skid Geometry

$$\begin{Bmatrix} W_D \\ W_L \\ W_N \end{Bmatrix} = \begin{bmatrix} \cos\eta & \beta\cos\eta & -\sin\eta \\ -\beta & 1 & 0 \\ \sin\eta & \beta\sin\eta & \cos\eta \end{bmatrix} [\gamma][\Gamma] \begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix} \qquad (3.2\text{-}94)$$

The component W_D lies along the element, the others at right angles to the element. The component W_L may be used to find the applied moment for the torsional motion in the gear. Only the component W_N contributes to the element rotation about the axle. It is assumed that the moment due to this force, ℓW_N, is balanced by the restraining moment $N(\eta, \dot{\eta})$;

$$\ell W_N + N(\eta, \dot{\eta}) = 0 \qquad (3.2\text{-}95)$$

It is assumed that the drag force on the trailing end is opposing the horizontal velocity of the end, and that its magnitude is the product of the vertical ground reaction with the coefficient of friction between the element and the ground. The derivation producing Eq. 3.2-44 is used here, with the pad velocity components replaced by the components of velocity of the element trailing edge, so that

$$\begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix} = V_G \begin{Bmatrix} \mu \dot{X}_e / |\dot{X}_e| \left(1 + (\dot{Y}_e/\dot{X}_e)^2\right)^{-1/2} \\ \mu \dot{Y}_e / |\dot{Y}_e| \left(1 + (\dot{X}_e/\dot{Y}_e)^2\right)^{-1/2} \\ 1 \end{Bmatrix} \qquad (3.2\text{-}96)$$

The Eqs. 3.2-94, 96 may be combined to yield W_N in terms of V_G :

$$W_N = V_G \, \Phi \qquad (3.2\text{-}97)$$

where

$$\Phi = \ell_1 \begin{Bmatrix} \sin\eta \\ \beta\sin\eta \\ \cos\eta \end{Bmatrix}' [\gamma]' [\Gamma] \begin{Bmatrix} \mu(\dot{X}_e/|\dot{X}_e|)\left[1 + (\dot{Y}_e/\dot{X}_e)^2\right]^{-1/2} \\ \mu(\dot{Y}_e/|\dot{Y}_e|)\left[1 + (\dot{X}_e/\dot{Y}_e)^2\right]^{-1/2} \\ 1 \end{Bmatrix} \qquad (3.2\text{-}98$$

This result may be substituted into Eq. 3.2-95 and the vertical ground reaction found in terms of the restraining moment. Use of Eq. 3.2-96 then yields the remaining components of the ground reaction. Thus, for $Z_A < 0$,

$$\begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix} = \frac{-N(\eta,\dot{\eta})\begin{Bmatrix} \mu(\dot{X}_e/|\dot{X}_e|)\left[1+(\dot{Y}_e/\dot{X}_e)^2\right]^{-1/2} \\ \mu(\dot{Y}_e/|\dot{Y}_e|)\left[1+(\dot{X}_e/\dot{Y}_e)^2\right]^{-1/2} \\ 1 \end{Bmatrix}}{\ell_1\begin{Bmatrix} \sin\eta \\ \beta\sin\eta \\ \cos\eta \end{Bmatrix}'[\gamma]'[\Gamma]\begin{Bmatrix} \mu(\dot{X}_e/|\dot{X}_e|)\left[1+(\dot{Y}_e/\dot{X}_e)^2\right]^{-1/2} \\ \mu(\dot{Y}_e/|\dot{Y}_e|)\left[1+(\dot{X}_e/\dot{Y}_e)^2\right]^{-1/2} \\ 1 \end{Bmatrix}} \qquad (3.2\text{-}99)$$

This result is quite similar to that obtained in the paragraph on surface pads for the case of rigid ground. The form is valid only until the element flattens out on the ground. Thereafter, the forces are derived exactly in the manner of the section on surface pads, except that the coefficient of friction will generally be different, and the components of the pad horizontal velocity become those of the element axle velocity. The axle height Z_A may be obtained by integrating the axle velocity normal to the ground, given by Eq. 2.10-6. The ground forces on the body, expressed in the body coordinate system, are then given by Eq. 3.2-35.

The element must be considered rigid in the previous derivation. If flexibility is to be included in the same direction as the component rigid body motion about a line, the problem may be approached in the same manner as that in Paragraph 4.2.5.2, which discusses articulated gears. The ground force is then defined in the following paragraphs.

Skis and Skids, Soft Ground

Prior to the time when the element flattens onto the ground, the ground forces may be derived from the general form for ground penetration forces expressed by Eq. 3.2-82, which is written in differential form as

$$dF = -\left(\frac{3}{2}\,\frac{B}{v}\right) v\, dA \qquad (3.2\text{-}100)$$

where the velocity v is that relative to the ground and normal to the differential area dA. The geometry is depicted in Fig. 7.

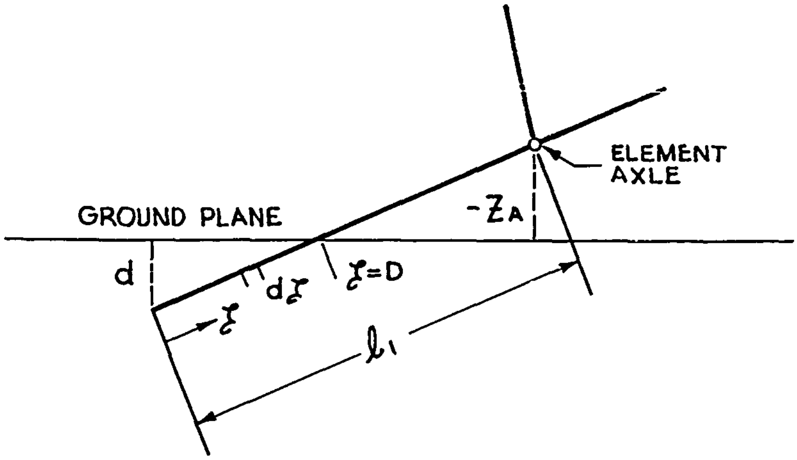


Figure 7. Ski or Skid Ground Penetration.

The parameter ζ is measured along the element from the trailing end; its value at the surface of the ground is D. This "planing" length is related to the penetration depth, d, the axle height, and the distance from the axle to the trailing end by

$$D = \left(\frac{d}{d - Z_A}\right)\ell_1 \tag{3.2-101}$$

The element is assumed to have a "planing" width w_1, which is independent of ζ, and a constant lateral width w_2. The forces normal to these areas are those in the contacting element coordinate system, defined in the paragraph on rigid ground. The differential force normal to an element of area $w_1 d\zeta$ is then

$$dW_N = -\left(\frac{3}{2}\frac{B}{v}\right) v_N(\zeta)\, w_1\, d\zeta \tag{3.2-102}$$

and that normal to an area $w_2 d\zeta$ is

$$dW_L = -\left(\frac{3}{2}\frac{B}{v}\right) v_L(\zeta)\, w_2\, d\zeta \tag{3.2-103}$$

The differential of friction force due to these forces is parallel to the element;

$$dW_D = \mu \left(dW_N + dW_L \right) \tag{3.2-104}$$

These three terms define the force on an elemental length $d\xi$. The velocities at that point are

$$v_N(\xi) = v_{NA} + \dot{\eta}(l_1 - \xi) \tag{3.2-105}$$

$$v_L(\xi) = v_{LA} + \dot{\beta} \cos \eta (l_1 - \xi) \tag{3.2-106}$$

The transformation from the gear coordinate system to that of the contacting element, expressed by Eq. 3.2-92, yields the components of axle velocity v_{NA}, v_{LA}. The axle velocity in the ground coordinate system is derived from Eq. 2.10-6, and may be transformed to the gear coordinate system. The desired axle velocities are given by

$$v_{NA} = \begin{Bmatrix} \sin \eta \\ \beta \sin \eta \\ \cos \eta \end{Bmatrix}' \left([\gamma]' \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + \begin{Bmatrix} \dot{P}_{x'} \\ \dot{P}_{y'} \\ \dot{P}_{z'} \end{Bmatrix}_A + [\Omega] \begin{Bmatrix} x' + P_{x'} \\ y' + P_{y'} \\ z' + P_{z'} \end{Bmatrix}_A \right) \tag{3.2-107}$$

$$v_{LA} = \begin{Bmatrix} -\beta \\ 1 \\ 0 \end{Bmatrix}' \left([\gamma]' \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + \begin{Bmatrix} \dot{P}_{x'} \\ \dot{P}_{y'} \\ \dot{P}_{z'} \end{Bmatrix}_A + [\Omega] \begin{Bmatrix} x' + P_{x'} \\ y' + P_{y'} \\ z' + P_{z'} \end{Bmatrix}_A \right) \tag{3.2-108}$$

in terms of the panel point displacement and velocity of the axle expressed in the gear coordinate system. These velocities are not dependent on the parameter ξ ; hence the forces are

$$W_N = -\left(\frac{3}{2}\frac{B}{v}\right)\omega_1 \left[v_{NA} + \dot{\eta}\left(\ell_1 - \frac{D}{2}\right)\right]D \qquad (3.2\text{-}109)$$

$$W_L = -\left(\frac{3}{2}\frac{B}{v}\right)\omega_2 \left[v_{LA} + \dot{\beta}\cos\eta\left(\ell_1 - \frac{D}{2}\right)\right]D \qquad (3.2\text{-}110)$$

$$W_D = \mu\left(W_N + W_L\right) \qquad (3.2\text{-}111)$$

in terms of the planing length D. From Eq. 3.2-101, the planing length is determined by the penetration depth. This may be obtained by integration of the velocity of the trailing end of the element normal to the ground. The addition of the velocity of the trailing end relative to the axle to the form in Eq. 2.10-6 yields the desired quantity. Thus,

$$d = \int_{t_0}^{t} \dot{Z}_e \, dt \qquad (3.2\text{-}112)$$

where

$$\dot{Z}_e = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}' [\Gamma]' \left[\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + [\gamma] \left(\begin{Bmatrix} \dot{P}_{x'} \\ \dot{P}_{y'} \\ \dot{P}_{z'} \end{Bmatrix}_A + \begin{bmatrix} \cos\eta & -\beta & \sin\eta \\ \beta\cos\eta & 1 & \beta\sin\eta \\ -\sin\eta & 0 & \cos\eta \end{bmatrix} \begin{Bmatrix} 0 \\ -\ell_1\dot{\beta}\cos\eta \\ \ell_1\dot{\eta} \end{Bmatrix} \right) \right.$$
$$\left. + [\Omega][\gamma] \left(\begin{Bmatrix} x' + P_{x'} \\ y' + P_{y'} \\ z' + P_{z'} \end{Bmatrix}_A + \ell_1 \begin{Bmatrix} 1 - \cos\eta \\ -\beta\cos\eta \\ \sin\eta \end{Bmatrix} \right) \right] \qquad (3.2\text{-}113)$$

Finally, the ground force expressed in the body coordinate system is

$$\begin{Bmatrix} Q_{xG} \\ Q_{yG} \\ Q_{zG} \end{Bmatrix}_m = [\gamma] \begin{bmatrix} \cos\eta & -\beta & \sin\eta \\ \beta\cos\eta & 1 & \beta\sin\eta \\ -\sin\eta & 0 & \cos\eta \end{bmatrix} \begin{Bmatrix} W_D \\ W_L \\ W_N \end{Bmatrix} \tag{3.2-114}$$

These forms are valid only until the element becomes horizontal, or parallel to the ground. It has been assumed that the element rotates about an axle, the rotation being described by the angle η. If this angle is allowed to vary, the forms above for ground forces will generally be valid until the axle height becomes zero. It is suggested that when this occurs, the forces may then be obtained by a form which assumes that the axle vertical velocity is zero. This is the form used in deriving the forces on a surface pad in the case of a rigid ground, which should hold in the case of a ski or skid of large surface area.

If the element is actually rigidly affixed to the gear, or the rotational spring-damper is sufficiently strong, the element will not become parallel to the ground and the above forms will remain valid.

3.2.5.5 Gas-Filled bags

Although this formulation of the landing impact problem does not describe gas-filledbag inertial properties, the bag and vehicle mass may be combined as a single rigid body to describe the inertia. At present, the only analytical work of any rigorous type has been restricted to vertical alightment. Ref. 15 is an example of this type of effort. In the notation of this report, the equation for vertical motion used there is

$$M \dot{v}_z = Q_z \tag{3.2-115}$$

The external forces considered in the reference are gravitational, atmospherical drag, and ground force from the gas-filled bag. The latter is given by

$$Q_{zG} = -A(P - P_o) \tag{3.2-116}$$

where P is the bag total pressure and P_o is the atmospheric pressure. The pressure variation in the bag is assumed adiabatic, for the case of no gas bleeding, so that

$$P = P_1 \left(\frac{V_1}{V}\right)^m \tag{3.2-117}$$

Then for a cylindrical, non-bulging bag, the volume is related linearly to the body deflection after touchdown;

$$V = V_1 \left(1 - \frac{d}{h}\right) \tag{3.2-118}$$

where h is the undeflected cylinder height and d is the deflection of the bag, obtained by integration of v_z from touchdown time.

Additional complications in the form used for the pressure in the bag will result if the bag shape varies or if a gas orifice is used. Likewise, the contacting area may be a variable. These points are discussed in the reference. It is pointed out that for stability requirements, a multiple bag system is usually helpful.

Generally, then, it is necessary to define the variation in the bag shape, contact area, and bag pressure in terms of the geometry of the bag, the bag deflection, and some polytropic compression form. The simplest approach is to assume that the bag shape remains constant. The contact area is then defined by the geometry and the bag vertical deflection. If the bag shape varies with pressure, the elastic properties of the bag must be included.

If the gas bleeding orifice is controlled such that the bag pressure remains constant, then the form for the vertical force is fairly simple. If the rate of bleeding does not produce this result, then the pressure variation must be described in some manner in terms of the bag deflection and deflection rate. The gas flow characteristics necessary to define the pressure variations will not be written here.

Suppose a set of independently operating bags are located on a plane surface on the vehicle. For small angles from the vertical, the restraining force from each bag may be considered to be nearly parallel to the body z-axis; each is given by

$$V_G = -A(P - P_o) \tag{3.2-119}$$

after touchdown, and is zero beforehand. The vertical height of the bags defines the touchdown times, as in the section on surface pods. The drag force on each bag is assumed to have components given by

$$D_G = \mu V_G \frac{\dot{X}_B}{|V_B|} \tag{3.2-120}$$

$$S_G = \mu V_G \frac{\dot{Y}}{|V_B|} \tag{3.2-121}$$

where μ is the coefficient of friction between the bag and the ground. Body forces are then given by Equation 3.2-34 and body moments by Equation 3.2-37. The instantaneous positions of the contacting points should be interpreted as the positions of the center points of the bag ends. The deflection P_y of each bag would be the stroking deflection, and the lateral deflections P_x, P_ψ may be derived from the drag forces and bag lateral spring rates. These spring rates will probably be dependent on vertical deflection. The vertical deflection of each bag will be found by integrating the velocity normal to the ground of the center point of the bag attachment area. The above statements are sufficient to work a stability problem for landing on a set of gas-filled bags, provided that the variation with all parameters of the bag area and pressure in Equation 3.2-116 are defined.

3.3 APPLIED FORCES ON COMPONENT RIGID BODY MOTION

3.3.1 General

The purpose of this section is to define the applied forces which enter Equations 2.8-18, 40 for component rigid body motion for the particular cases of gear piston stroking and bogie rotational motion. The forces are of two types, ground forces and stroking forces.

For gear stroking, the total force along the piston axis is given by

$$Q_{z'P} = Q_{z'G} + Q_{z'S} \tag{3.3-1}$$

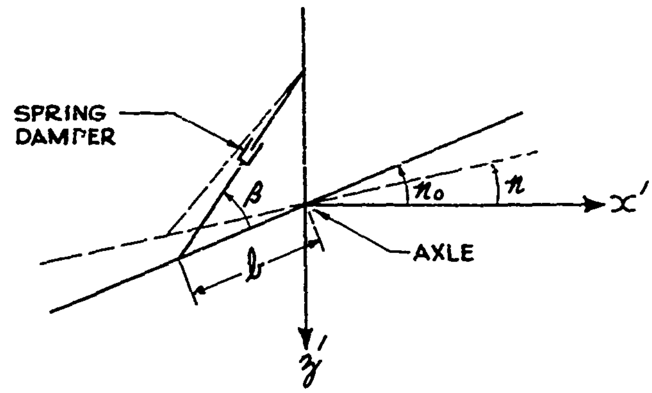
where $Q_{z'G}$ is the component of the ground force along the stroking axis, and $Q_{z'S}$ is the total stroking force.

For bogie rotation, the total moment is due to the ground forces and the rotational damper forces. The ground forces are derived later. The damper moment may be expressed by

$$N = N(\eta, \dot{\eta}) \tag{3.3-2}$$

where η is the bogie rotational angle. The relation between this angle and its time derivative to the actual damping mechanism is considered briefly here. If the damping mechanism is a stroking element affixed to the bogie element and the gear piston, the exact relation will be non-linear. The gear coordinate axes

have been translated



to the bogie axle for convenience. The bogie is originally at an angle η_0 from the x -axis, defining the position of zero stroke for the rotational spring damper. If the bogie rotates only through a small angle, the restraining moment is related to the spring damper force F_{SD} by

$$N = b \sin\beta \, F_{SD}$$

The force may be derived from the paragraphs or springs and damping mechanisms by replacing the element stroke by

$$s = (b \sin\beta)(\eta - \eta_0)$$

for small rotations. Hence

$$\dot{s} = (b \sin\beta)\dot{\eta}$$

For large rotations, these expressions must be replaced by forms nonlinear in the angle η .

3.3.2 Ground Forces

The Equation 3.2-35 defines the force on the body due to interaction with the ground of a single contacting element. This force may be transformed into the gear coordinate system to yield the components needed for gear stroking and bogie rotation. The force on a single contacting element is written in the body coordinate system as

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}_{G_m} = [\Gamma] \begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix}_m \qquad (3.3\text{-}3)$$

This is written in the gear coordinate system as

$$\begin{Bmatrix} Q_{x'} \\ Q_{y'} \\ Q_{z'} \end{Bmatrix}_{G_m} = [\gamma]' [\Gamma] \begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix}_m \qquad (3.3\text{-}4)$$

If there are several contacting elements on a single gear, this form is summed over those elements;

$$\begin{Bmatrix} Q_{x'} \\ Q_{y'} \\ Q_{z'} \end{Bmatrix}_G = [\gamma]' [\Gamma] \sum_m \begin{Bmatrix} D_G \\ S_G \\ V_G \end{Bmatrix}_m \qquad (3.3\text{-}5)$$

This form yields the ground forces in the gear coordinate system. The forces $Q_{x'G_m}$, $Q_{z'G_m}$ are used in the calculation of bogie moments, and the force $Q_{z'G}$ is used to define the ground force for piston stroking.

3.3.3 Stroking Forces

The purpose of this paragraph is to define the stroking force $Q_{z'S}$. This force is the summation of all the forces on the piston, along the stroking axis, except for ground forces. It is important to realize that the formulation is sufficiently general that the effects of any known type of energy-absorbing mechanism may be included simply by defining the stroking force properly.

The remainder of Section 3.3 will be devoted to the stroking forces. Obviously, not all the various types of shock-absorbing devices can be considered. Those to be included are as follows. Paragraph 3.3.3.1 covers hydraulic forces. Variations with oil compressibility, metering pins, and relief valves are included. In Paragraph 3.3.3.2, springs of the mechanical, pneumatic, and liquid types are considered. Paragraph 3.3.2.3 discusses bearing friction forces for a particular configuration. The piston bottoming or restraining forces are defined in Paragraph 3.3.3.4. Crushable materials are discussed in Paragraph 3.3.3.5, and gas expulsion

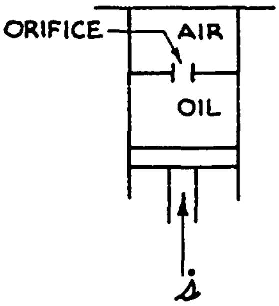
devices in Paragraph 3.3.3.6 complete the types of stroking forces to be developed in the report.

In order to write the relations defining these forces in compact form, the displacement of the piston relative to the cylinder will be used. This displacement, referred to as stroke and given the symbol s , is measured from the fully extended position and is positive as the piston enters the cylinder. The equation governing the piston motion is written in terms of the piston displacement relative to the body coordinate system, $\Delta_{z'}$, which is positive along the z'-axis of the gear coordinate system. Let the point at which the cylinder is affixed to the vehicle be labeled B , and its total displacement $\mathbb{P}_B$. These three variables are then related by

$$s = -\Delta_{z'} + \mathbb{P}_B \cdot \mathbb{K}' = -\Delta_{z'} + P_{z'B} \qquad (3.3\text{-}6)$$

This result indicates that stroking may occur due to elastic deformations in the vehicle. This result must be used in all the forcing functions in this section, if the vehicle is considered to be flexible. The stroking velocity may be obtained by taking the time derivative of Equation 3.3-6.

3.3.3.1 Hydraulic force, F_H



When the piston shown in the figure is displaced with a velocity $\dot{s}$, a pressure field is created in the lower chamber which resists this motion. In a rigorous definition of this pressure field, where consideration is given to viscosity and compressibility of the fluid and the unsteady aspects of the flow, one is faced with a formidable and in most cases unsolvable formulation. Satisfactory definitions of loads in the piston have been achieved, however, by semi-empirical means where the form for the pressure in the piston is defined by Bernoulli's principle for an ideal fluid. Neglecting the static head, the velocity of the jet stream at a point outside the orifice at which the streamlines are parallel is given by:

$$v = \sqrt{\dot{s}^2 + \frac{2\overline{P}}{\rho_H} - \frac{2\overline{P}_a}{\rho_H}} \qquad (3.3\text{-}8)^*$$

where

$\dot{s}$ = stroking velocity of the piston
$\overline{P}$ = pressure upstream of the orifice where the streamlines are parallel
$\overline{P}_a$ = pressure in the airchamber
ρ_H = density of the fluid

*Equations on this page misnumbered.

In actual practice the velocity v is never attained due to the dissipation of energy in overcoming the resistance to flow. The ratio of the actual velocity to the velocity given by the Bernoulli's relation is defined by the coefficient of velocity, C_V. Equating the rate at which the volume of fluid is displaced by the piston to the rate at which it is discharged through the orifice,

$$A_H \dot{s} = A_j C_V v \tag{3.3-9}$$

where A_H is the hydraulic area of the piston, and A_j is the area of the jet stream at a point where the streamlines are parallel. The ratio of A_j to the orifice area A_N is given by the coefficient of contraction, C_C. For most hydraulic dampers, $\dot{s}$ is small compared to v, and combining Eqs. 3.3-7, 8 yields

$$A_H{}^2 \dot{s}^2 = \frac{2(C_C C_V)^2 A_N{}^2}{\rho_H} (\bar{P} - \bar{P}_a) \tag{3.3-10}$$

The hydraulic force F_H is then given by:

$$F_H = (\bar{P} - \bar{P}_a) A_H = \frac{\rho_H A_H{}^3 \dot{s}^2}{2(C_D A_N)^2} \tag{3.3-11}$$

where the orifice coefficient C_D is the product of C_C and C_V.

The value of the coefficient C_D is dependent upon the size of the orifice, the shape and finish of the orifice face, the kinematic viscosity of the fluid, the velocity, and the motion in the fluid approaching the orifice which causes a dependence on the length of the oil column remaining in the strut. The relationship between size, velocity and kinematic viscosity can be expressed as Reynold's number. Thus, the value of the orifice coefficient for a given orifice can then be said to be dependent upon Reynold's number and piston stroke.

The results of an experimental study of orifice coefficients in a small oleo-pneumatic strut with a constant orifice are contained in Ref. 16. This test investigated the effect on orifice coefficient of variations of Reynold's number in the range from 9,500 to 66,500. The results of the test were summarized by an empirical relation between orifice coefficient, stroke and stroke velocity. The variation between the minimum and maximum values of the coefficient for the tests ranged from 0.86 to 0.93. The final conclusion from the experiment was that an average value of orifice coefficient could be used as a constant to determine strut loads.

Effect of Oil Compressibility

When the contact velocity of the strut is high (generally above 15 feet per second), the compressibility of the hydraulic oil and in certain cases the volumetric expansion of the strut will have an effect on the load time history. The following treatment of this effect has improved substantially the correlation between analysis and drop test results at Chance Vought.

The change in pressure $d\bar{P}$ in the fluid due to compressing the volume by an amount dv is given by

$$d\bar{P} = K\frac{dv}{V} \tag{3.3-12}$$

where K would be an equivalent bulk modulus expressing both the compressibility of the fluid and the elasticity of the strut. Since the volume change is small compared to the total volume, the differentials can be replaced by finite differences

$$\Delta\bar{P} = K\frac{\Delta V}{V} \tag{3.3-13}$$

The term Δv is the difference between the volume swept by the hydraulic area and the volume expelled through the orifice:

$$\frac{d(\Delta V)}{dt} = A_H \dot{s} - C_D A_N v \tag{3.3-14}$$

For a constant A_H,

$$\Delta V = A_H s - \int_0^t C_D A_N v \, dt \tag{3.3-15}$$

The volume V is the volume of the oil chamber at any time, hence

$$V = V_0 - A_H s \tag{3.3-16}$$

where V_0 is the volume of the oil chamber when the strut is fully extended. Thus, $\Delta\bar{P}$ becomes

$$\Delta\bar{P}=K\frac{A_H s - \int_0^t C_D A_N v\, dt}{V_c - A_H s} \tag{3.3-17}$$

The term $\Delta\bar{P}$ represents the total change in pressure from time $t=0$ to time "t," so that

$$\Delta\bar{P} = \bar{P}-\bar{P}_{A_0} \tag{3.3-18}$$

where $\bar{P}$ is the pressure in the oil chamber and $\bar{P}_{A_0}$ is the initial air pressure. The pressure $\Delta\bar{P}$ is due to hydraulic pressure $\bar{P}_H$ and air pressure $\bar{P}_A$;

$$\Delta\bar{P} = \bar{P}_H + \bar{P}_A - \bar{P}_{A_0} \tag{3.3-19}$$

thus the expression for $\bar{P}_H$ becomes

$$\bar{P}_H = K\frac{A_H s - \int_0^t C_D A_N v\, dt}{V_0 - A_H s} - \bar{P}_A + \bar{P}_{A_0} \tag{3.3-20}$$

The velocity v is defined as the Bernoullian velocity,

$$v = \sqrt{\frac{2\bar{P}_H}{\rho_H}} \tag{3.3-21}$$

The hydraulic force F_H is given by

$$F_H = A_H \bar{P}_H \tag{3.3-22}$$

Orifice and Relief Valve Combination

The relief valve operates on the principle that when the hydraulic pressure reaches a predetermined value, a valve cracks, introducing additional orifice area. Theoretically, this method of controlling the load in the strut is superior to the metering pin approach since it functions from load level rather than stroke. There are, however, sufficient design, qualification and manufacturing difficulties

to detract from its theoretical superiority.

Writing a general expression for a hydraulic strut employing a relief valve is difficult since designs vary widely. The following is offered as an example to illustrate the concept. The geometry is depicted in Fig. 8.

The piston of area A_H strokes with a velocity $\dot{s}$, metering fluid through an orifice with area A_4. When the pressure $\bar{P}_1$ acting on the area A_1 overcomes the downward force F_{R0} on the valve due to the spring, the valve moves upward, uncovering an area $A_3(x_R)$. The pressure acting on the relief valve before cracking is $\bar{P}_H$. The force required for cracking is therefore $\bar{P}_H A_1$.

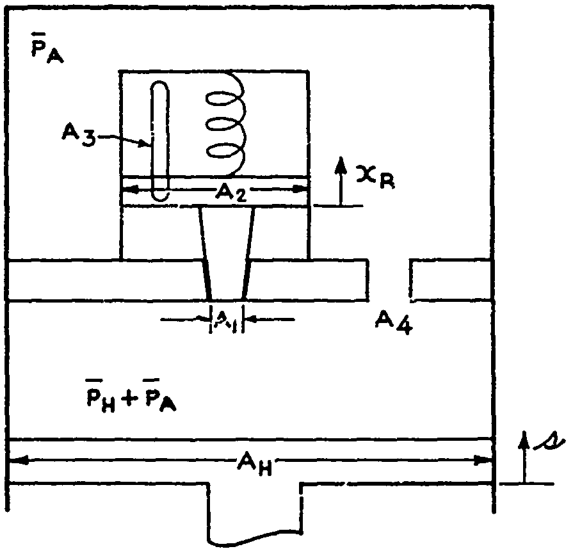


Figure 8. Hydraulic Damper with Relief Valve.

The force acting on the valve due to the spring is given by

$$F_R = F_{R_0} + K_R x_R \tag{3.3-23}$$

where K_R is the spring rate of the relief valve spring.

If the compressibility of the fluid and the mass of the relief valve can be neglected,

$$\bar{P}_H = \frac{\rho_H A_H^2}{2C_D^2 A_4^2}\,\dot{s}^2 \qquad \bar{P}_H A_1 < F_{R_0} \tag{3.3-24}$$

$$\bar{P}_H = \frac{\rho_H A_H^2 \dot{s}^2}{2C_D^2\left[A_4 + A_3(x_R)\right]^2} \qquad \bar{P}_H A \geqslant F_{R_0} \tag{3.3-25}$$

$$A_3(x_R) \text{ LIMIT VALUE} = A_i$$

where

$$x_R = \frac{\bar{P}_H A_2 - F_{R_0}}{K_R} \tag{3.3-26}$$

The hydraulic force F_H is then

$$F_H = A_H \bar{P}_H \tag{3.3-27}$$

Oil compressibility effects can be included in the same manner as discussed previously.

Extension Stroke

In general the damping characteristics of a self-positioning strut made up of a spring and hydraulic damper in series will be different during the extension stroke than during compression. The extension characteristics are governed by two requirements: oil must be returned to lower chamber fairly rapidly for another energy absorption cycle; and adequate damping must be provided to reduce bottoming loads at the zero stroke position during rapid extension.

The energy available for the extension stroke is that stored in the gear spring which may be pneumatic, liquid or mechanical.

The form for the hydraulic force is the same as for the compression stroke except the constants are different

$$F_H = -\frac{\rho_H A_{H5}^3}{2(C_D A_N)_5^2}\,\dot{s}^2 \tag{3.3-28}$$

where the parameters are

A_{HS} = hydraulic area associated with the extension stroke
$(C_D A_N)_S$ = effective orifice area associated with the extension stroke

The relations for the hydraulic force previously derived are summarized below. The relation defining the stroke, s, from Eq. 3.3-6 should be kept in mind.

Incompressible oil, metering pin

$$F_H = \frac{\rho_H A_H^3 \dot{s}^2}{2(C_D A_N)^2} \qquad \dot{s} \geqslant 0 \tag{3.3-29}$$

$$F_H = -\frac{\rho_H A_{HS}^3 \dot{s}^2}{2(C_D A_N)_S^2} \qquad \dot{s} < 0 \tag{3.3-30}$$

Incompressible oil, relief valve

$$F_H = \frac{\rho_H A_H^3 \dot{s}^2}{2(C_D A_4)^2} \qquad \begin{matrix} \dot{s} \geqslant 0 \\ \bar{P}_H A_1 \geqslant F_{R_0} \end{matrix} \tag{3.3-31}$$

$$F_H = \frac{\rho_H A_H^3 \dot{s}^2}{2 C_D^2 [A_4 + A_3(x_4)]^2} \qquad \begin{matrix} \dot{s} \geqslant 0 \\ \bar{P}_H A_1 \geqslant F_{R_0} \end{matrix} \tag{3.3-32}$$

$$x_R = \frac{\bar{P}_H A_2 - F_{R_0}}{K_R} \tag{3.3-33}$$

$$F_H = -\frac{\rho_H A_{HS}^3 \dot{s}^2}{2(C_D A_N)_S^2} \qquad \dot{s} < 0 \tag{3.3-34}$$

Compressible oil, metering pin

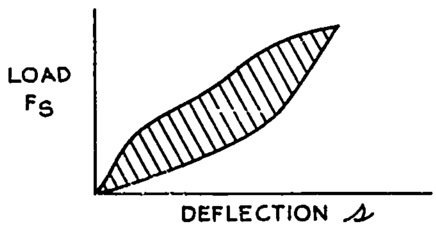
$$\left.\begin{aligned} F_H &= \bar{P}_H A_H \\ \bar{P}_H &= K \frac{A_H s - \int_0^t C_D A_N v \, dt}{V_0 - A_H s} - \bar{P}_A + \bar{P}_{A0} \\ v &= \sqrt{\frac{2\bar{P}_H}{\rho}} \end{aligned}\right\} \quad \dot{s} \geqslant 0 \qquad (3.3\text{-}35)$$

$$F_H = \frac{-\rho_H A_{HS}^3 \dot{s}^2}{2(C_D A_N)_S^2} \qquad \dot{s} < 0 \qquad (3.3\text{-}36)$$

The parameter A_N is in each case the metering function, which may be a function of stroke, and may be of a form for the return stroke different from that of the compression stroke.

3.3.3.2 Spring forces

A spring is by definition an elastic body or device that returns to its original shape after being distorted. A spring when distorted will generate a restoring force that is functionally related to the displacement of the spring along a specified axis. If energy is dissipated during the distortion-recovery cycle the functional relationship between restoring force and deflection will be double valued, as shown below, where the shaded portion represents the energy dissipated:



The general expression for a spring exhibiting the above load-deflection characteristic is given by

$$\begin{aligned} F_S &= f_{S_1}(s) \qquad & \dot{s} \geqslant 0 \\ F_S &= f_{S_2}(s) \qquad & \dot{s} < 0 \end{aligned} \qquad (3.3\text{-}37)$$

Since the spring itself has inertia, the load-deflection characteristics are dependent somewhat on the rate of loading of the spring. These effects, in general, will be small for the anticipated applications of this report and have not been considered.

Three types of springs are considered in the succeeding paragraphs: mechanical, pneumatic, and liquid.

Mechanical Springs

Mechanical springs occur in almost endless variety, from simple helical springs, which possess linear load-deflection characteristics, to the more complicated Belleville and ring springs which exhibit non-linear characteristics and significant energy dissipation. Mechanical springs lend themselves to fairly accurate analytical description since they will be deformed, in general, only in the elastic region of the material. Detailed formulas of spring load-deflection characteristics, however, will not be derived in this report since these relations are well documented in design manuals, texts and manufacturers' literature.

For the purposes of this report it will be assumed that the force-deflection characteristics for a mechanical spring can be expressed as:

$$\begin{aligned} F_S &= f_{S1}(\delta) \qquad && \dot{\delta} \geqslant 0 \\ F_S &= f_{S2}(\delta) && \dot{\delta} < 0 \end{aligned} \tag{3.3-38}$$

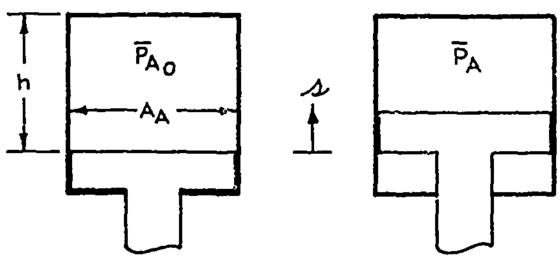
For springs in which hysteresis is negligible $f_{S1} = f_{S2}$ and the cycle is defined by a single function.

Assuming the functional relation for F_S is known, δ must still be expressed in terms of the variables of the analysis, as in Eq. 3.3-6.

Pneumatic Spring

A pneumatic spring is one that depends upon the compressibility of a gas to generate a restoring force. The law governing the compressibility of a gas in a closed container is given by

$$\overline{P}_{A0}\ V_0^{\,n} = \overline{P}_A\ V^{\,n} = \text{CONSTANT} \tag{3.3-39}$$



where

$\bar{P}_{A0}$ ~ initial pressure in cylinder
V_0 ~ initial volume
$\bar{P}_A$ ~ pressure at stroke "Δ"
V ~ volume at stroke "Δ" $= V_0 - A_A \Delta$
n ~ exponent that indicates the exact polytropic nature of the compression
A_A ~ pneumatic area

Substituting the relation for V into the above equation the expression for $\bar{P}_A$ becomes

$$\bar{P}_A = \frac{\bar{P}_{A0} V_0^{\,n}}{(V_0 - A_A \Delta)^n} \tag{3.3-40}$$

and the force F_A acting downward on the piston is

$$F_A = A_A \left(\bar{P}_A - P_a \right) \tag{3.3-41}$$

The atmospheric pressure P_a may usually be neglected in comparison with the cylinder pressure.

The value of the exponent n to be used depends on how much heat is transferred to and from the gas. If the compression or expansion takes place rapidly, such that little heat is transferred from and to the gas, the process can be assumed adiabatic and n becomes the ratio of the specific heat of the gas at constant pressure to the specific heat at constant volume. For dry air $n = 1.406$. If the compression or expansion process is such that the temperature of the gas is unchanged (isothermal), $n = 1$. In the general, polytropic case, n must be determined from an analysis of the thermodynamic process.

When the pneumatic spring discussed above is part of an oleo-pneumatic strut, the thermodynamic process is further complicated by the cooling action and vaporization of the oil spray. The net effect of this spray is to cause the thermodynamic process to approach isothermal. Experiments to evaluate the exponent n for an oleo-pneumatic strut, reported in Ref. 17, indicate an average value of $n = 1.06$ would adequately represent the compression process for the impacts investigated. For most practical analyses it is sufficiently accurate to choose $n = 1$, for which case

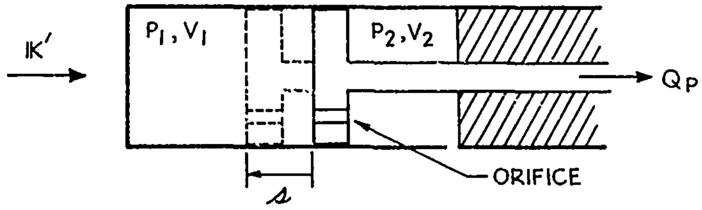
$$F_A = \frac{\bar{P}_{A0} V_0}{\frac{V_0}{A_A} - \Delta} \tag{3.3-42}$$

Here again the variable "s" must be written in terms of the variable of the analysis, as given by Eq. 3.3-6.

Liquid Spring

In recent years a number of aircraft landing gears have been designed solely on hydraulic principles, where the function of the pneumatic spring is replaced by compressing the hydraulic fluid. Usually some mechanical advantage is involved between wheel stroke and shock absorber stroke since the liquid spring-shock absorber is inherently a short stroke device. Two spring and damper concepts were examined. Although these differ materially from a design and utility standpoint, they both submit to the same analytical treatment.

A simple geometry for a liquid spring is pictured below. Initially, the volume V_{10} to the left of the piston and the volume V_{20} to the right of the piston are filled with a compressible fluid to some initial pressure. (The mechanical stops are not pictured.)



The force on the piston is made up of the hydraulic force resulting from the difference in hydraulic pressure and area on either side of the orifice, the frictional force resulting from the normal pressure of the seals, and a force Q_P which represents any other forces external to the liquid spring system. The piston force F_P is therefore:

$$F_P = F_H + F_F + Q_A$$

$$F_H = P_1 A_P - P_2 (A_P - A_R)$$
$$= A_P (P_1 - P_2) + A_R P_2$$

where A_P is the area of the piston and A_R is the area of the rod.

The frictional force F_F will generally be a significant contribution to the net force in the piston due to the severe sealing requirements.

The analytical expressions for the normal forces on the piston due to the sealing pressure will be functions of the particular design. In general they can be expressed as:

$$F_{NU} = f_U(P_1, P_2)$$

$$F_{NL} = f_L(P_1, P_2)$$

The frictional force on the piston is then:

$$F_F = (\mu_U F_{NU} + \mu_L F_{NL}) \frac{\dot{s}}{|\dot{s}|} \qquad \text{FOR } \mu_U F_{NU} + \mu_L F_{NL} < Q_A + F_H \tag{3.3-43}$$

and

$$F_F = -(Q_A + F_H) \qquad \text{FOR } \mu_U F_{NU} + \mu_L F_{NL} \geqslant Q_A + F_H \tag{3.3-44}$$

where μ_U and μ_L are the friction coefficients of the upper and lower seal respectively.

The bulk moduli of oils used in liquid springs in general vary linearly with pressure throughout the range of interest. Thus, in the two regions,

$$B_1 = a + b P_1 = V_1 \frac{dP_1}{dV_1} \tag{3.3-45}$$

$$B_2 = a + b P_2 = V_2 \frac{dP_2}{dV_2} \tag{3.3-46}$$

Assume for the moment that the contribution to the change in total volume from the cylinder and seal elasticity can be neglected. The instantaneous volumes of oil on either side of the piston are dependent on both stroke and the amount of fluid which has been metered through the orifice. These will be given by

$$V_1 = V_{10} - A_P s + V_M \tag{3.3-47}$$

$$V_2 = V_{20} + (A_P - A_R) s - V_M \tag{3.3-48}$$

and their differentials by

$$dV_1 = -A_P\, ds + dV_M \tag{3.3-49}$$

$$dV_2 = (A_P - A_R)\, ds - dV_M \tag{3.3-50}$$

The volume of oil metered through the orifice is related to the orifice coefficient C_D, the orifice area A_N, and the velocity of the oil in the orifice v, by

$$dV_M = C_D A_N v\, dt \tag{3.3-51}$$

where the Bernoullian velocity,

$$v = \sqrt{\frac{2(P_1 - P_2)}{\rho_H}} \tag{3.3-52}$$

is used. With these expressions, Eq. 3.3-35 becomes

$$a + b P_1 = \frac{\left(V_{10} - A_P s + \int_0^t C_D A_N v\, dt\right) dP_1}{-A_P\, ds + C_D A_N v\, dt} \tag{3.3-53}$$

which may be written as

$$\dot{P}_1 = \frac{(a + b P_1)(-A_P \dot{s} + C_D A_N v)}{V_{10} - A_P s + \int_0^t C_D A_N v\, dt} \tag{3.3-54}$$

A similar form may be developed from Eq. 3.3-46. The resulting expressions which define the force acting on the piston rod are recapitulated below.

Liquid Spring, Rigid Cylinder and Seals

$$\left.\begin{aligned}
F_P &= A_P(P_1 - P_2) + A_R P_2 + F_F + Q_P \\
\dot{P}_1 &= \frac{(a + b P_1)(-A_P \dot{s} + C_D A_N v)}{V_{10} - A_P s + \int_0^t C_D A_N v \, dt} \\
\dot{P}_2 &= \frac{(a + b P_2)[(A_P - A_R)\dot{s} - C_D A_N v]}{V_{20} + (A_P - A_R)s - \int_0^t C_D A_N v \, dt} \\
v &= \sqrt{\frac{2(P_1 - P_2)}{\rho_H}}
\end{aligned}\right\} \qquad (3.3\text{-}55)$$

The inclusion of the volumetric expansion of each region due to cylinder expansion and seal compression is quite complicated. The change in volume from each of these effects must be written in terms of the pressure and the piston stroke. In terms of the geometrical volume change dV_{1C} due to cylinder expansion in region one and dV_{1S} due to compression of any seal in region one, and similar terms in region two, the above relations may be restated as follows.

Liquid Spring, Elastic Cylinder and Seals

$$\left.\begin{aligned}
F_P &= A_P(P_1 - P_2) + A_R P_2 + F_F + Q_P \\
\dot{P}_1 &= \frac{(a + b P_1)\left[-A_P \dot{s} + C_D A_N v + \frac{\partial V_{1C}}{\partial P_1}\dot{P}_1 + \frac{\partial V_{1C}}{\partial s}\dot{s} + \frac{\partial V_{1S}}{\partial P_1}\dot{P}_1\right]}{V_{10} - A_P s + \int_0^t C_D A_N v \, dt} \\
\dot{P}_2 &= \frac{(a + b P_2)\left[(A_P - A_R)\dot{s} - C_D A_N v + \frac{\partial V_{2C}}{\partial P_2}\dot{P}_2 + \frac{\partial V_{2C}}{\partial s}\dot{s} + \frac{\partial V_{2S}}{\partial P_2}\dot{P}_2\right]}{V_{20} + (A_P - A_R)s - \int_0^t C_D A_N v \, dt} \\
v &= \sqrt{\frac{2(P_1 - P_2)}{\rho_H}}
\end{aligned}\right\} \qquad (3.3\text{-}56)$$

Thus, if the cylinder is considered inelastic, the corresponding partial derivatives may be set equal to zero. Generally, each of the partial derivatives will be a function of pressure, stroke, or both. The terms in seal compression are constant if the seals are compressed in the linear range; otherwise they will vary with pressure.

The geometry of a particular liquid spring may be somewhat different from that presented here, but the approach will remain the same. The metering function A_N is retained underneath the integral signs in the previous forms, as it may vary with stroke and hence implicitly with time. Relief valves for rapid return strokes have not been included here. They may be incorporated by simply stating that the pressures in the two regions are identical.

3.3.3.3 Bearing friction force, F_F

When the piston strokes within the cylinder an amount of kinetic energy will be dissipated by friction at the bearing surfaces between the piston and the cylinder. The frictional force, opposing the motion of the piston, is expressed as

$$F_F = \left[\mu_{BL} \left| F_{BL} \right| + \mu_{BU} \left| F_{BU} \right| + F_{FO}\right] \frac{\dot{s}}{|\dot{s}|} \tag{3.3-57}$$

where

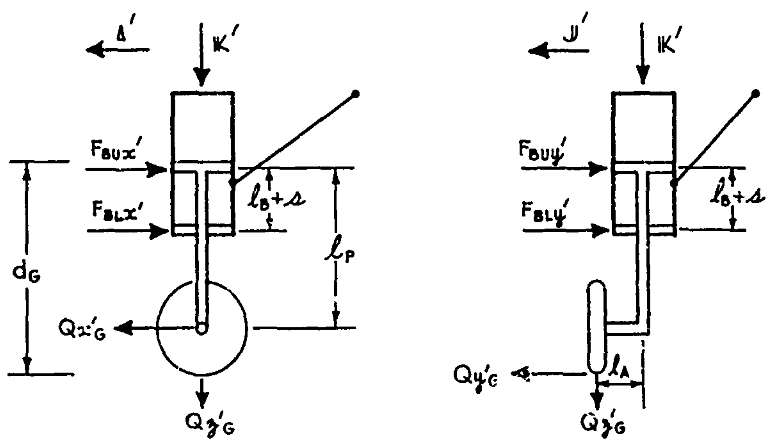
F_{BU}, F_{BL} ~ bearing forces at the upper and lower bearings respectively required to balance the lateral loading on the piston.

μ_{BU}, μ_{BL} ~ coefficients of sliding friction at the upper and lower bearings respectively.

F_{FO} ~ frictional force at zero lateral loading. This force is assumed to be known.

The bearing forces F_{BU} and F_{BL} are the reactions that put the piston in equilibrium with the external and inertial forces in the piston.

Consider first the case when the inertia forces in the piston can be neglected when determining bearing forces.



From the equilibrium conditions on the piston;

$$F_{BLx'} = \frac{Q_{x'G}\, \ell_P}{\ell_B + \Delta} \tag{3.3-58}$$

$$F_{BUx'} = Q_{x'G}\left(1 - \frac{\ell_P}{\ell_B + \Delta}\right)$$

$$F_{BLy'} = \frac{-Q_{z'G}\, \ell_A + Q_{y'G}\, d_G}{\ell_B + \Delta} \tag{3.3-59}$$

$$F_{BUy'} = Q_{y'G}\left(1 - \frac{d_G}{\ell_B + \Delta}\right) + Q_{zG}\, \ell_A$$

where ℓ_B is the bearing separation at zero stroke.

The resultant bearing forces are therefore,

$$F_{BL} = \sqrt{F_{BLx'}^2 + F_{BLy'}^2} \tag{3.3-60}$$

$$F_{BU} = \sqrt{F_{BUx'}^2 + F_{BUy'}^2} \tag{3.3-61}$$

The dimension d_G is the length of the piston, ℓ_p, plus the tire radius, $\bar{r}$, minus the tire deflection along the $\mathbb{K}$ axis, $\delta \cos\alpha_A$, or

$$d_G = \ell_p + \bar{r} - \delta \cos \alpha_A \tag{3.3-62}$$

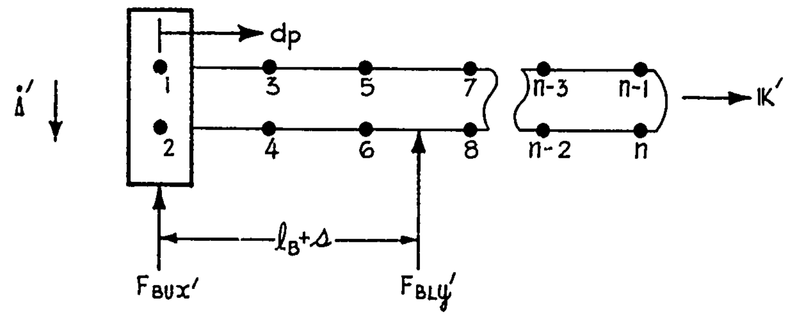
where δ and α_A are defined by Equation 3.2-55.

In the general case when the inertial loads are to be considered, the panel point loads on the piston are defined from the equation of motion of the piston

$$\{F_{Px'}\} = [K_{x'x'}]\{P_{x'}^{e}\} \tag{3.3-63}$$

$$\{F_{Py'}\} = [K_{y'y'}]\{P_{y'}^{e}\} \tag{3.3-64}$$

where $[K_{x'x'}]$, $[K_{y'y'}]$ are the stiffness matrices of the piston and $\{P_{x'}^{e}\}$, $\{P_{y'}^{e}\}$ are the elastic displacements of the piston.



Given the loads at the above-indicated panel points the bearing forces are determined from the equations of statics as before:

$$F_{BLx'} + F_{BUx'} = \{1\}'\{F_{Px'}\} \tag{3.3-65}$$

$$F_{BLx'}\,(\ell_B + \Delta) = \{d_P\}'\{F_{Px'}\} \tag{3.3-66}$$

where $\{d_P\}$ is the distance of each panel point to $F_{BUx'}$ along $\mathbb{K}'$.

Thus

$$F_{BLx'} = \frac{\{d_P\}' [K_{x'x'}] \{P_{x'}^{\,e}\}}{\ell_B + s} \tag{3.3-67}$$

and

$$F_{BUx'} = \frac{(\{1\}'(\ell_B + s) - \{d_P\}') [K_{x'x'}] \{P_{x'}^{\,e}\}}{\ell_B + s} \tag{3.3-68}$$

In the same manner,

$$F_{BLy'} = \frac{\{d_P\}' [K_{y'y'}] \{P_{y'}^{\,e}\}}{\ell_B + s} \tag{3.3-69}$$

$$F_{BUy'} = \frac{(\{1\}'(\ell_B + s) - \{d_P\}') [K_{y'y'}] \{P_{y'}^{\,e}\}}{\ell_B + s} \tag{3.3-70}$$

The resultant bearing forces are again given by

$$F_{BL} = \sqrt{F_{BLx'}^{\,2} + F_{BLy'}^{\,2}} \tag{3.3-71}$$

$$F_{BU} = \sqrt{F_{BUy'}^{\,2} + F_{BUx'}^{\,2}} \tag{3.3-72}$$

3.3.3.4 Bottoming force, F_B

The piston of the hydraulic strut will be required to stay within certain values of stroke, consistent with the construction of the strut. For strokes less than zero the piston will contact the lower bearing. For strokes greater than δ_{MAX} the piston will contact some mechanical stop at the upper end of the cylinder. To ensure that the piston stays within the required range of strokes the following functions are introduced.

$$F_B = K_{BL}\,\delta \qquad\qquad \delta \leqslant 0 \tag{3.3-73}$$

$$F_B = K_{BU}\,\delta \qquad\qquad \delta \geqslant \delta_{MAX} \tag{3.3-74}$$

where K_{BU}, K_{BL} are the spring rates of the upper and lower ends of the cylinder.

The values of the spring rates are quite high, and may arbitrarily be assigned such that the displacement past the mechanical stop is extremely small. Generally, damping terms are included so that the piston will cease to oscillate from the bottoming and stroking spring forces. This is a mathematical artifice, and the damper rates are arbitrary.

3.3.3.5 Crushable materials

The use of crushable materials in shock-mitigating devices is fairly recent. There is, however, a wide range of types of these devices. The design of such a device will depend greatly on the vehicle mass and the limits to be placed on its deceleration rate.

Almost all of the shock-absorbing mechanisms using crushable materials are designed so that the force on the main component never exceeds a certain value but remains very near that value throughout the gear stroke. This is possible since the primary characteristic of most crushable materials is that of a constant load-stroke curve. This generally holds until the volume of the material is reduced to one-fifth of its initial value, at which time the crushing characteristics become nonlinear.

Some shock-absorbing devices must have the characteristic that the rate of loading does not exceed a certain value, rather than the load itself. This is accomplished by shaping the leading end of the device which penetrates the material. Most of the crushable materials will exhibit a high onset force, which quickly reduces to the constant force for which it is designed. This is the reason for the shaping of the initially crushed surface or the device which does the crushing.

It has been assumed here that the crushable material is interior to a stroking device. If it is a shaped piece of material simply affixed to the underside of a vehicle, the force should be considered as an exterior applied force on the body rather than an interior force applied to an unspring mass.

If the material is interior, the contribution would be positive;

$$F_C = \begin{cases} 0 & s, \dot{s} < 0 \\ +f_0 & s, \dot{s} \geqslant 0 \end{cases} \tag{3.3-75}$$

and if it is exterior, the force is a ground force applied to the vehicle at touchdown time t_0, and is negative,

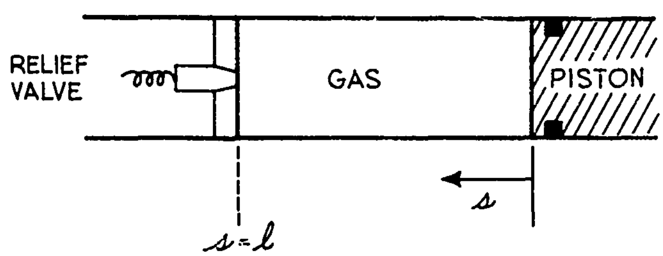
$$V_{GC} = \begin{cases} 0 & t < t_0 \\ -f_0 & t \geqslant t_0 \end{cases} \tag{3.3-76}$$

If it is exterior, the contribution to V_G from ground flexibility is usually negligible. The drag force would depend on the vertical load and a coefficient of friction dependent on the surface area of the material on the ground.

3.3.3.6 Gas compression and relief valves

This paragraph is concerned with the energy absorption device which makes use of gas compression and release through an orifice.

A device of this type is usually designed with a pressure-sensitive bleeding mechanism. The orifice area opened by this instrument is designed such that the interior pressure is maintained as constant as possible. Consider the figure below.



The spring holding the relief valve is preloaded so that a critical pressure P_C is required to open the valve. Prior to the opening of the valve, the pressure inside the cylinder is found from the usual polytropic form in terms of the initial pressure and volume;

$$PV^n = P_o V_o^{\,n} \tag{3.3-77}$$

The volume of the gas during this interval is linearly related to the stroke, so

that

$$P = P_o \left(1 - \frac{s}{\ell}\right)^{-n} \tag{3.3-78}$$

In terms of the area A_v of the valve prior to opening and the spring preloading force f , the critical pressure at which the valve will open is

$$P_c = A_v f \tag{3.3-79}$$

The corresponding critical stroke is

$$s_c = \ell \left[1 - \left(\frac{P_o}{P_c} \right)^{\frac{1}{n}} \right] \tag{3.3-80}$$

After the relief valve is opened, it is assumed that the area of the opening may vary sufficiently rapidly that the cylinder pressure remains at the critical value. The force on the piston is then

$$F = \begin{cases} A P_o \left(1 - s/\ell\right)^{-n} & s \leq s_c \\ A P_c & s > s_c \end{cases} \tag{3.3-81}$$

in terms of the area A of the piston.

The assumption has been made that the piston stroking velocity is sufficiently low that the pressure is uniform throughout the cylinder.

3.4 ELASTIC BODY FORCES

3.4.1 General

The forces and internal reactions giving rise to elastic deformations in the vehicle are discussed in this paragraph. Perhaps the most difficult point to

understand in the formulation of the equations of motion for an elastic body is in this area. Several different concepts may be used in the interpretation of the panel point equations. These concepts vary according to the manner in which the internal reactions in the vehicle are entered into the equations governing component elastic motions. The reactions may be entered entirely as constraints on the elastic motions, or they may be entered partially as applied loads. The former of these methods is perhaps the most straightforward. It does not, however, lend itself readily to approximate solutions, as the transformation to modal coordinates is not easily accomplished with that method. The latter will then be used. The method will be elaborated here, with several examples which exhibit the concepts.

Consider a wing attached in a cantilevered manner to a fuselage; that is, it does not rotate relative to the fuselage. The reactions at the wing root which hold the wing to the fuselage are not considered to be applied forces or constraints on the wing. Their effect is entered into the stiffness matrix for the wing, so that the stiffness matrix is that of a cantilevered wing. Thus, the equations defining the wing modal coordinates in the paragraph on modal transformations will then produce cantilevered mode shapes for the wing. Consider the symbolic form for the wing panel point equations;

$$\left[A_W\right]\left\{\ddot{P}_W\right\} + \left[K_W\right]\left\{P_W^{e}\right\} = \left\{\hat{Q}_W\right\} \qquad (3.4\text{-}1)$$

where the stiffness matrix is that of a cantilevered wing, and the right-hand side includes both the applied forces and terms coupling rigid body motion and elastic motion. There are no constraints on the wing. If the fuselage elastic displacements are set equal to zero, the wing total displacement becomes the wing elastic displacement, and the left-hand side of the equation becomes the form which defines the wing modal coordinates. The equation defining the wing panel point total displacements is

$$\left\{P_W\right\} = \left\{P_W^{e}\right\} + \left[T_{WF}\right]\left\{P_F\right\} \qquad (3.4\text{-}2)$$

The matrix $\left[T_{WF}\right]$ relates the displacements of the wing panel points - with the wing considered as a rigid body - due to displacements of the fuselage panel points. This matrix then picks out only the fuselage panel points located at the wing root, and geometrically defines the wing displacements due to fuselage displacements. The displacements of a rigid body are completely defined by the displacements of any three points in the body, so that three panel points at the wing root are sufficient for the general case.

This geometric relationship has a useful property. Suppose Eq. 3.4-2 is substituted into Eq. 3.4-1, and the latter is premultiplied by the transform of the matrix $\left[T_{WF}\right]$. Rearrangement produces

$$\left[T_{WF}\right]'\left[K_W\right]\left\{P_W^{e}\right\} = \left[T_{WF}\right]'\left(\left\{\hat{Q}_W\right\} - \left[A_W\right]\left\{\ddot{P}_W^{e}\right\}\right) - \left[T_{WF}\right]'\left[A_W\right]\left[T_{WF}\right]\left\{\ddot{P}_F\right\} \qquad (3.4\text{-}3)$$

The latter term on the right-hand side is just the inertial load on the fuselage panel points at the wing root due to their acceleration of the "rigid" wing. The other terms on the right-hand side are loads on those panel points due to wing elastic accelerations and wing externally applied forces. Consider the symbolic form for the fuselage elastic motion;

$$[A_F]\{\ddot{P}_F\}+[K_F]\{P_F\}=\{\hat{Q}_F\} \tag{3.4-4}$$

where again the right-hand side includes applied forces and forces due to coupling of the rigid body and fuselage elastic motions. One of the applied forces on the fuselage elastic motion is at the wing root. This force is represented in the column matrix form by either side of Eq. 3.4-3. Then Eq. 3.4-4 may be written

$$\begin{aligned}[A_F]\{\ddot{P}_F\}+[K_F]\{P_F\}=\{\hat{Q}_F\}^{*}+[T_{WF}]'\left(\{\hat{Q}_W\}-[A_W]\{\ddot{P}_W{}^{e}\}\right)\\ -[T_{WF}]'[A_W][T_{WF}]\{\ddot{P}_F\}\end{aligned} \tag{3.4-5}$$

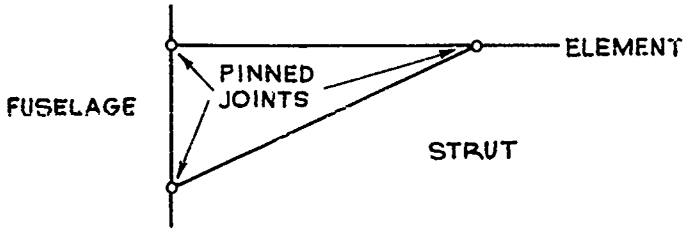
where the starred (*) quantity does not include the wing root applied forces, and the right-hand side of Eq. 3.4-3 has been used. The reason that the left side was not used is now obvious, for transposition of the last term on the right to the left-hand side introduces the inertial effects of the "rigid" wing into the total mass matrix needed in the definition of the fuselage modes from Eq. 2.9-12;

$$[A]=[A_F]+2\,[T_{WF}]'[A_W][T_{WF}]+\cdots\cdots \tag{3.4-6}$$

In a like manner, the applied forces on the fuselage panel points from the various appended components may be evaluated. Thus, it is seen in this example that the wing root forces are applied forces on the fuselage elastic motion, but their effect on the wing elastic motion is included in the definition of the wing stiffness matrix.

This approach will normally be used throughout the analysis; the reaction forces are applied forces on the more central component and are included in the stiffness matrix for the more remote component. Thus, for a gear affixed to a wing, the trunnion loads are applied forces on the wing elastic motion, but their effect on the gear is included in the gear stiffness matrix. It is apparent that if a chain of components affixed to one another are all elastic, this may become a lengthy process.

In Section 4 it is pointed out that, for practical applications, some of the components would be considered to be rigid. Suppose that for a particular application, the fuselage may be considered to be rigid. There would then be no fuselage panel point displacements. Consider an element which is pin-supported to the fuselage, and has a pin-supported strut connected to the fuselage. This configuration is pictured below.

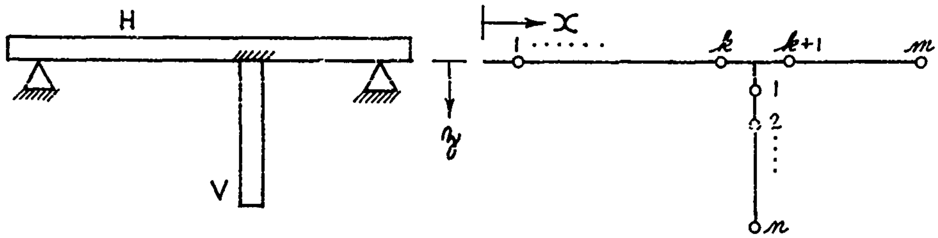


Assume that the strut is rigid, and that the fuselage is rigid. The elastic motion of the element is defined by its distributed stiffness properties and the two restraints. For this geometry, the elastic deflection of the element must be zero at the two support points. This effect will be included in the stiffness matrix of the element; the stiffness matrix is that of an element simply supported at two points. In this case there are no applied loads on fuselage elastic motion.

Let the fuselage now be elastic, so that the support points may move. The strut is still assumed to be rigid. The motion of the panel points on the fuselage where the element is supported obviously determines the displacement of the element as a rigid body. Thus, the problem becomes exactly the same in concept as the first case of a fuselage with cantilevered wing. The stiffness matrices of this element and of the wing differ due to the change in support methods, and the geometric relations involved in the matrix $[T_{WF}]$ are different from those in this case. The applied forces on the fuselage panel points are found in the identical manner.

These examples show the manner in which the internal reactions will be handled. Cases have been discussed in which both components are elastic, and in which one component is rigid. If both components are assumed to be rigid, but are attached to a third elastic component, the same methods apply in obtaining applied forces on the elastic component. The panel point elastic displacements of the components assumed to be rigid are simply set equal to zero. The proper geometric properties are then used to enter the applied forces on the elastic component due to the inertia of, and applied forces on, the rigid components.

An example of the general method is shown for the simplified geometry below. The horizontal beam is simply supported at two points, and the vertical beam is cantilevered to the horizontal beam. Motion is considered only in the plane of the page, and each beam is assumed to be incompressible.



The horizontal (H) beam may have elastic displacements parallel to the z axis. The vertical (V) beam may have elastic displacements parallel to the x axis, and motion as a rigid body parallel to both axes due to elastic displacements of the horizontal beam. The total displacements on the vertical beam are

$$\begin{Bmatrix}\{P_x\}\\ \{P_z\}\end{Bmatrix}_V = \begin{Bmatrix}\{P_x{}^e\}\\ \{\ 0\ \}\end{Bmatrix}_V + [T_{VH}]\{P_z\}_H \qquad (3.4\text{-}7)$$

Assume that the interpolation scheme used to calculate the mass matrices is the trapezoidal scheme; i.e., the displacement between panel points is a straight line ending at the panel point displacements. The panel points labeled k, $k+1$ on the horizontal beam then determine entirely the motion of the vertical beam as a rigid body. The displacements of the panel points on the vertical beam are then geometrically related to those of the panel points k, $k+1$ by the matrix $[T_{VH}]$, which yields Eq. 3.4-8. It is assumed that these two points are spaced a distance $d/2$ to either side of the connection point. The distances z_1, z_2, ..., z_n are measured from the connection point down to each panel point on the vertical beam.

$$
\begin{Bmatrix} \begin{Bmatrix} P_{x_1} \\ P_{x_2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_{x_m} \end{Bmatrix} \\ \begin{Bmatrix} P_{z_1} \\ P_{z_2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_{z_m} \end{Bmatrix} \end{Bmatrix}_V = \begin{Bmatrix} \begin{Bmatrix} P_{x_1}^e \\ P_{x_2}^e \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_{x_m}^e \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \end{Bmatrix}_V + \begin{bmatrix} 0, \cdots, 0, & z_1/d, & -z_1/d, & 0, \cdots, 0 \\ 0, \cdots, 0, & z_2/d, & -z_2/d, & 0, \cdots, 0 \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ 0, \cdots, 0, & z_m/d, & -z_m/d, & 0, \cdots, 0 \\ 0, \cdots, 0, & 1/2, & 1/2, & 0, \cdots, 0 \\ 0, \cdots, 0, & 1/2, & 1/2, & 0, \cdots, 0 \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ \cdot \cdots \cdot & \cdot & \cdot & \cdot \cdots \cdot \\ 0, \cdots, 0, & 1/2, & 1/2, & 0, \cdots, \end{bmatrix} \begin{Bmatrix} P_{z_1} \\ \vdots \\ P_{z_k} \\ P_{z_{k+1}} \\ \vdots \\ P_{z_m} \end{Bmatrix}_H \qquad (3.4\text{-}8
$$

The stiffness matrix for the horizontal beam is that of a beam simply supported at two points. The stiffness matrix for the vertical beam is that of a beam cantilevered at one end. The motion of the entire system as a rigid body does not occur due to the fixed simple supports on the horizontal beam. The corresponding rigid body coupling terms from the panel point Eqs. 2.8-5 may then be omitted. No constraints are formally entered into the motion, so that the resulting panel point equations may be written as

$$
[A_{zz}]_H \{\ddot{P}_z\}_H + [K_{zz}]_H \{P_z\}_H = \{Q_z\}_H \qquad (3.4\text{-}9
$$

$$
\begin{bmatrix} [A_{xx}] & \\ & [A_{zz}] \end{bmatrix}_V \begin{Bmatrix} \{\ddot{P}_x\} \\ \{\ddot{P}_z\} \end{Bmatrix}_V + \begin{Bmatrix} [K_{xx}]\{P_x^e\} \\ \{R\} \end{Bmatrix}_V = \begin{Bmatrix} \{Q_x\} \\ \{Q_z\} \end{Bmatrix}_V \qquad (3.4\text{-}1
$$

The quantity $\{R\}$ is discussed in a moment. For simplicity, assume that the only external applied forces on the system occur at the bottom panel point on the vertical beam. Let this force have the components F_x , F_z . The applied force on the vertical beam is then

$$
\{Q_x\}_V = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ F_x \end{Bmatrix} \qquad \{Q_z\}_V = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ F_z \end{Bmatrix} \qquad (3.4\text{-}1
$$

The applied force on the horizontal beam is due only to the vertical beam connection. This is given by

$$\{Q_z\}_H = [T_{V_H}]'\{R\} = [T_{V_H}]'\left(\begin{Bmatrix}\{Q_x\}\\ \{Q_z\}\end{Bmatrix}_V - \begin{bmatrix}[A_{xx}] & \\ & [A_{zz}]\end{bmatrix}_V \begin{Bmatrix}\{\ddot{P}_x\}\\ \{\ddot{P}_z\}\end{Bmatrix}_V\right) \tag{3.4-12}$$

The quantity $\{R\}$ is seen to be the difference in the applied force and inertial reaction of the vertical beam along its axis. This quantity cannot be determined from stiffness properties, as it was assumed that the beam is rigid along the axis. The stiffness in that direction is then infinite, but the elastic deformation is zero, and the product is indeterminant. Thus, $\{R\}$ is found in terms of the applied forces and inertial reactions.

The first term on the right-hand side of Eq. 3.4-12 is easily evaluated;

$$[T_{V_H}]'\begin{Bmatrix}\{Q_x\}\\ \{Q_z\}\end{Bmatrix}_V = \begin{Bmatrix}0\\ \vdots\\ 0\\ (z_m/\,d)F_x + \tfrac{1}{2}F_z\\ -(z_m/\,d)F_x + \tfrac{1}{2}F_z\\ 0\\ \vdots\\ 0\end{Bmatrix} \tag{3.4-13}$$

This form enters the effect of the applied forces on the vertical beam into the horizontal beam panel points k, $k+1$ in the proper manner. The other term on the right-hand side of Eq. 3.4-12 essentially subtracts the inertial effects of the vertical beam from the effects of the applied forces on the vertical beam. In that term, the Eq. 3.4-7 will be substituted. The two resulting terms are

$$[T_{V_H}]'\begin{bmatrix}[A_{xx}] & \\ & [A_{zz}]\end{bmatrix}_V \begin{Bmatrix}\{\ddot{P}_x^{e}\}\\ \{0\}\end{Bmatrix}_V + \left([T_{V_H}]'\begin{bmatrix}[A_{xx}] & \\ & [A_{zz}]\end{bmatrix}_V [T_{V_H}]\right)\{\ddot{P}_z\}_H \tag{3.4-14}$$

The latter will be transposed to the left-hand side of Eq. 3.4-9 and included with the first term there. This term enters the inertial effects of the vertical beam as a rigid body into the proper place in the mass matrix of the horizontal beam. The remaining term enters the inertial effects of the lateral elastic

motion of the vertical beam as applied forces on the panel points k , $k+1$. The equations of motion are then

$$\left([A_{zz}]_H + [T_{VH}]'[A_V][T_{VH}]\right)\{\ddot{P}_z\}_H + [K_{zz}]_H\{P_z\}_H \tag{3.4-15}$$

$$= \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ (z_n/d)F_x + \tfrac{1}{2}F_z \\ -(z_n/d)F_x + \tfrac{1}{2}F_z \\ 0 \\ \vdots \\ 0 \end{Bmatrix} - [T_{VH}]' \begin{bmatrix} [A_{xx}] & \\ & [A_{zz}] \end{bmatrix}_V \begin{Bmatrix} \{\ddot{P}_x^e\} \\ \{0\} \end{Bmatrix}_V$$

$$[A_{xx}]_V\{\ddot{P}_x\}_V + [K_{xx}]_V\{P_x^e\}_V = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ F_x \end{Bmatrix} \tag{3.4-16}$$

The equation governing motion of the vertical beam parallel to the z-axis could be included, but no additional information is obtained. This motion is defined entirely by the matrix $[T_{VH}]$ and the displacements of the panel points k , $k+1$. If this is kept in mind, the equation may be omitted. The form of Eq. 3.4-15 is now proper for the transformation to modal coordinates, as the mass matrix includes the rigid body inertia of the vertical beam. The Eq. 3.4-16 is in the proper form when the total displacement is expanded according to Eq. 3.4-7.

This completes the discussion of the applied forces on component motions. The forces on the whole vehicle, which were discussed in Paragraph 3.2, must be distributed to the various panel points. This process will not be discussed here, as it depends only on the particular configuration. Several examples in the next section exhibit this process.

Two types of applied forces on panel point elastic motion are explicitly written next. One is the aerodynamic force on a wing due to changes in the local surface angle of attack caused by wing elastic deformations. The other is the structural damping force, which is physically not an external force, but is considered to be as a convenience.

3.4.2 Panel Point Aerodynamic Forces

The contribution to individual panel point loads here will be considered only on airplane wings, and only that contribution due to elastic motions. The body and wing z axes will be assumed coincident. The pressure distribution due to elastic deformations is

$$P^{e}(x',y',t) = L_R(x',y',t) + \frac{1}{v} L_c(x',y',t) \qquad (3.4\text{-}17)$$

where

$$L_R(x',y',t) = \frac{1}{2}\rho v^2 \frac{\partial C_p(x',y')}{\partial \alpha} \frac{\partial P_{z'}(x',y',t)}{\partial x'} \qquad (3.4\text{-}18)$$

$$L_c(x',y',t) = \frac{1}{2}\rho v^2 \frac{\partial C_p(x',y')}{\partial \alpha} \dot{P}_{z'}(x',y',t) \qquad (3.4\text{-}19)$$

and

ρ = atmospheric density
v = velocity of the vehicle relative to the atmosphere
α = airplane angle of attack
C_p = local pressure coefficient

The virtual work entailed in a change of displacement $\delta P_{z'}$ is

$$\delta W = \int_{W_1+W_2} P^{e}(x',y',t)\, \delta P_{z'}\, dx'dy' \qquad (3.4\text{-}20)$$

where the integration is over both wings.

This may be written in terms of an interpolation scheme as

$$\delta W = \left\{\delta P_{z'}\right\}_{W_1} \frac{1}{2}\rho v^2 \left(\left[L_R\right]_{W_1}\left\{P_{z'}\right\}_{W_1} + \frac{1}{v}\left[L_c\right]_{W_1}\left\{\dot{P}_{z'}\right\}_{W_1}\right) \qquad (3.4\text{-}21)$$

$$+ \left\{\delta P_{z'}\right\}_{W_2} \frac{1}{2}\rho v^2 \left(\left[L_R\right]_{W_2}\left\{P_{z'}\right\}_{W_2} + \frac{1}{v}\left[L_c\right]_{W_2}\left\{\dot{P}_{z'}\right\}_{W_2}\right)$$

Panel point loads for either wing are then

$$\left\{Q_{z'A}^{\ e}\right\} = \frac{1}{2}\rho v^2\left(\left[L_R\right]\left\{P_{z'}\right\} + \frac{1}{v}\left[L_C\right]\left\{\dot{P}_{z'}\right\}\right) \tag{3.4-22}$$

It has been assumed that $\left\{Q_{x'A}\right\}$ and $\left\{Q_{y'A}\right\}$ are unchanged from their rigid values.

The contributions to body forces and moments from these panel point forces are

$$Q_{zA}^{\ e} = \{1\}'\left\{Q_{z'A}^{\ e}\right\}_{W_1} + \{1\}'\left\{Q_{z'A}^{\ e}\right\}_{W_2} \tag{3.4-23}$$

$$N_{xA}^{\ e} = \{y\}'_{W_1}\left\{Q_{z'A}^{\ e}\right\}_{W_1} + \{y\}'_{W_2}\left\{Q_{z'A}^{\ e}\right\}_{W_2} \tag{3.4-24}$$

$$N_{yA}^{\ e} = \{x\}'_{W_1}\left\{Q_{z'A}^{\ e}\right\}_{W_1} + \{x\}'_{W_2}\left\{Q_{z'A}^{\ e}\right\}_{W_2} \tag{3.4-25}$$

These may be of significant value for large wings with considerable bending.

The inclusion of contributions from tails or other surfaces may be performed in a similar manner but will not have any first-order effect on landing loads.

3.4.3 Structural Damping

Damping in a built-up structure is not a problem to be approached analytically. The results of damping are easily incorporated for systems of defined normal modes, however.

In the test laboratory, the definition of a mode of vibration includes the requirement that the shape of the deformation remain constant as the amplitude decreases to zero. This may be used to define a damping parameter in the modal equations. Consider the equation

$$\left[A\right]\left\{\ddot{P}\right\} + \left[K_N\right]\left\{P\right\} = \{0\} \tag{3.4-26}$$

where $\left[K_N\right]$ is the stiffness matrix which defines the normal modes of vibration. The transformation to normalized modal coordinates yields

$$\left[\diagdown 1 \diagdown\right]\left\{\ddot{q}\right\} + \left[\diagdown \frac{1}{\lambda} \diagdown\right]\left\{q\right\} = \{0\} \tag{3.4-27}$$

a separated set of equations. The frequency of the undamped motion defined by each equation is $\omega = \frac{1}{\sqrt{\lambda}}$; the equation governing that coordinate is

$$\ddot{q} + \omega^2 q = 0 \tag{3.4-28}$$

Suppose that damping were included in some manner such that

$$[A]\{\ddot{P}\} + [D]\{\dot{P}\} + [K_N]\{P\} = \{0\} \tag{3.4-29}$$

If transformation to the (normalized) normal modes is made, the equation becomes

$$\{\ddot{q}\} + [\Phi]'[D][\Phi]\{\dot{q}\} + [{}^{1}/_{\lambda}]\{q\} = \{0\} \tag{3.4-30}$$

The requirement that the shape of the deformation remain unchanged as the motion dies out requires that the equations for the modal coordinates be uncoupled even with damping, which means that

$$[\Phi]'[D][\Phi] = [d] \tag{3.4-31}$$

where the elements are the modal damping coefficients. These elements may be assigned values in terms of critical damping, but are not derivable from basic properties of the structure in most cases. Consider the equation

$$\ddot{q} + d\dot{q} + \omega^2 q = 0 \tag{3.4-32}$$

where ω is the frequency of vibration of the mode corresponding to q for the undamped structure. If the substitution

$$q = e^{-\frac{d}{2}t} f(t) \tag{3.4-33}$$

is made, the equation for $f(t)$ is

$$\ddot{f} + (\omega^2 - d^2/4) f = 0 \tag{3.4-34}$$

The solution is harmonic, of frequency

$$\omega' = \sqrt{\omega^2 - d^2/4} \tag{3.4-35}$$

Critical damping is defined as that value for which $\omega' = 0$, that is,

$$d_c = 2\omega \tag{3.4-36}$$

If the fraction of critical damping for the mode is ζ_ω , or

$$d = \zeta_\omega d_c \tag{3.4-37}$$

and if $\zeta_\omega \ll 1$, the frequency of the damped motion is very nearly that of the undamped motion. Thus the damping term may be written

$$d = 2\omega \zeta_\omega \tag{3.4-38}$$

For many applications, the fraction of critical damping of the basic or fundamental mode of vibration may be taken as ten percent.

3.5 CONSTRAINTS

The reader who has progressed through the contents of the report to this point is well aware of the important part the concept of constraints may have in the formulation of a complex problem. The purpose here is to discuss the general concept of a constraint and to present some examples which may aid the reader.

In the formulation of a complex problem, it is often quite difficult to retain compactness and at the same time choose a set of independent variables which completely define the motion. It is more often convenient to choose a set of variables which least complicate the entire problem and use the method of constraints to eliminate the dependencies which have been included. There are also, in some cases, problems in which the applied forces corresponding to the independent variables cannot be defined, but may be found as constraint forces using a set of dependent variables. These concepts are discussed in Ref. 1. In that reference, it is shown that the constraints must be of a particular form in order that they may be handled by the methods presented here.

On the right-hand side of Eq. 2.8-5 is a set of terms

$$\sum_{j} \sigma_j \begin{Bmatrix} \left\{\frac{\partial F_j}{\partial P_{x'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{y'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{z'}}\right\} \end{Bmatrix}_i \qquad (3.5\text{-}1)$$

The functions F_j are the constraint relations; they are algebraic relations between some of the panel point displacements. These algebraic relations must be of a particular form. They must relate only the variables already defined in the set of equations of motion; they cannot introduce new variables. They must be written as

$$F_j\left(P_{x'k}, P_{x'l}, \cdots, P_{x'm}, t\right) = 0 \qquad (3.5\text{-}2)$$

where $k, l, \cdots, m$ indicate the particular panel point displacements that are not independent. Note that this form does not include inequalities.

The elements σ_j in Eq. 3.5-1 are called Lagrange's undetermined multipliers. They are the factors which convert each partial derivative into a constraint force consistent with the constraint relations. These elements are determined only by solution of the equations of motion, and they are generally complicated functions of the panel point motion throughout the system. They may, however, be eliminated before solving the equations for many cases of interest in this report.

A simple problem to illustrate the constraint concept is pictured in Fig. 9.

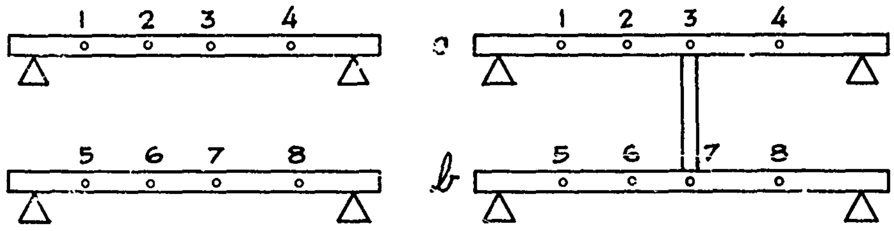


Fig. 9. A Constraint Example

In the left part, two beams are simply supported at two points. Each beam is assigned four panel points, and each beam is then allowed four degrees of freedom for elastic motion. The equations of motion are

$$[A]_a\{\ddot{P}\}_a + [K]_a\{P\}_a = \{Q\}_a \tag{3.5-3}$$

$$[A]_b\{\ddot{P}\}_b + [K]_b\{P\}_b = \{Q\}_b \tag{3.5-4}$$

where

$$\{P\}_a = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}_a \quad , \quad \{P\}_b = \begin{Bmatrix} P_5 \\ P_6 \\ P_7 \\ P_8 \end{Bmatrix}_b \tag{3.5-5}$$

There are eight degrees of freedom altogether. Suppose now a rigid, weightless bar is connected between the beams at panel points three and seven. These two panel points are now constrained to move together or have the same displacements. This constraint relation is written as

$$F_1 = P_3 - P_7 = 0 \tag{3.5-6}$$

which is of the general form of Eq. 3.5-2. The partial derivatives of this constraint relation with respect to the panel point displacements are

$$\begin{Bmatrix} \frac{\partial F_1}{\partial P_1} \\ \frac{\partial F_1}{\partial P_2} \\ \frac{\partial F_1}{\partial P_3} \\ \frac{\partial F_1}{\partial P_4} \end{Bmatrix}_a = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} \frac{\partial F_1}{\partial P_5} \\ \frac{\partial F_1}{\partial P_6} \\ \frac{\partial F_1}{\partial P_7} \\ \frac{\partial F_1}{\partial P_8} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{Bmatrix} \tag{3.5-7}$$

The Eqs. 3.5-3, 4 are then no longer valid; the constraint term must be added. The equations of motion which must be solved simultaneously for the coupled elastic motion are then

$$[A]_a \begin{Bmatrix} \ddot{P}_1 \\ \ddot{P}_2 \\ \ddot{P}_3 \\ \ddot{P}_4 \end{Bmatrix}_a + [K]_a \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \sigma_1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} \tag{3.5-8}$$

$$[A]_b \begin{Bmatrix} \ddot{P}_5 \\ \ddot{P}_6 \\ \ddot{P}_7 \\ \ddot{P}_8 \end{Bmatrix} + [K]_b \begin{Bmatrix} P_5 \\ P_6 \\ P_7 \\ P_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\sigma_1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix} \tag{3.5-9}$$

$$P_3 - P_7 = 0 \tag{3.5-10}$$

There are now the eight panel point displacements and one Lagrangian undetermined multiplier or nine unknown variables and nine equations of motion. On the right-hand side are the eight panel point applied forces and the constraint forces at panel points three and seven which cause those panel points to move together.

The reader whose experience with these concepts is limited should now examine in detail the Sections 2.8, 9 in order to obtain some working knowledge on the subject. The section on component rigid body motion along a line follows directly from the simple example, except that the number of constraints involved is large. Fortunately, in that and the following cases of use of the constraint methods, the undetermined multipliers may be eliminated. The constraint forces for each case are simply the forces necessary to hold the component together to move as a rigid body under the applied and inertial forces.

APPENDIX A

DERIVATION OF THE KINETIC ENERGY

The details of the derivation of the kinetic energy as presented in Section 2 are written here. Several vector and matrix identities are established for later use.

A vector in three-space may be written in matrix form as

$$\mathbb{A} = \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}' \begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix} = \begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix}' \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} \tag{A-1}$$

A scalar product is then written as

$$\begin{aligned} \mathbb{A}\cdot\mathbb{B} &= \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}' \begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix} \cdot \begin{Bmatrix} \mathbb{I} \\ \mathbb{J} \\ \mathbb{K} \end{Bmatrix}' \begin{Bmatrix} B_x \\ B_y \\ B_z \end{Bmatrix} \\ &= \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}' \begin{bmatrix} \mathbb{I}\cdot\mathbb{I} & \mathbb{I}\cdot\mathbb{J} & \mathbb{I}\cdot\mathbb{K} \\ \mathbb{J}\cdot\mathbb{I} & \mathbb{J}\cdot\mathbb{J} & \mathbb{J}\cdot\mathbb{K} \\ \mathbb{K}\cdot\mathbb{I} & \mathbb{K}\cdot\mathbb{J} & \mathbb{K}\cdot\mathbb{K} \end{bmatrix} \begin{Bmatrix} B_x \\ B_y \\ B_z \end{Bmatrix} \\ &= \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} B_x \\ B_y \\ B_z \end{Bmatrix} \\ &= \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}' \begin{Bmatrix} B_x \\ B_y \\ B_z \end{Bmatrix} \end{aligned} \tag{A-2}$$

In the same manner, the vector product is written as

$$\mathbb{A}\times\mathbb{B} = \begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}' \begin{Bmatrix}\mathbb{i}\\ \mathbb{J}\\ \mathbb{K}\end{Bmatrix} \times \begin{Bmatrix}\mathbb{i}\\ \mathbb{J}\\ \mathbb{K}\end{Bmatrix} \begin{Bmatrix}B_x\\B_y\\B_z\end{Bmatrix} \tag{A-3}$$

$$= \begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}' \begin{bmatrix}0 & \mathbb{K} & -\mathbb{J}\\ -\mathbb{K} & 0 & \mathbb{i}\\ \mathbb{J} & -\mathbb{i} & 0\end{bmatrix} \begin{Bmatrix}B_x\\B_y\\B_z\end{Bmatrix}$$

and the triple scalar product as

$$\mathbb{A}\cdot\mathbb{B}\times\mathbb{D} = \begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}' \begin{Bmatrix}\mathbb{i}\\ \mathbb{J}\\ \mathbb{K}\end{Bmatrix} \cdot \begin{Bmatrix}B_x\\B_y\\B_z\end{Bmatrix}' \begin{bmatrix}0 & \mathbb{K} & -\mathbb{J}\\ -\mathbb{K} & 0 & \mathbb{i}\\ \mathbb{J} & -\mathbb{i} & 0\end{bmatrix} \begin{Bmatrix}D_x\\D_y\\D_z\end{Bmatrix}$$

$$= \begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}' \begin{bmatrix}B_x\mathbb{i}\,\cdot & B_y\mathbb{i}\,\cdot & B_z\mathbb{i}\,\cdot\\ B_x\mathbb{J}\,\cdot & B_y\mathbb{J}\,\cdot & B_z\mathbb{J}\,\cdot\\ B_x\mathbb{K}\,\cdot & B_y\mathbb{K}\,\cdot & B_z\mathbb{K}\,\cdot\end{bmatrix} \begin{bmatrix}0 & \mathbb{K} & -\mathbb{J}\\ -\mathbb{K} & 0 & \mathbb{i}\\ \mathbb{J} & -\mathbb{i} & 0\end{bmatrix} \begin{Bmatrix}D_x\\D_y\\D_z\end{Bmatrix} \tag{A-4}$$

$$= \begin{Bmatrix}A_x\\A_y\\A_z\end{Bmatrix}' \begin{bmatrix}0 & -B_z & B_y\\ B_z & 0 & -B_x\\ -B_y & B_x & 0\end{bmatrix} \begin{Bmatrix}D_x\\D_y\\D_z\end{Bmatrix}$$

A coordinate transformation may be written in this manner:

$$\begin{Bmatrix}\mathbb{i}\\ \mathbb{J}\\ \mathbb{K}\end{Bmatrix} = [\gamma] \begin{Bmatrix}\mathbb{i}'\\ \mathbb{J}'\\ \mathbb{K}'\end{Bmatrix}$$

Then

$$[\gamma] = \begin{Bmatrix} \mathbf{i} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix} \cdot \begin{Bmatrix} \mathbf{i}' \\ \mathbf{J}' \\ \mathbf{K}' \end{Bmatrix} = \begin{bmatrix} \mathbf{i} \cdot \mathbf{i}' & \mathbf{i} \cdot \mathbf{J}' & \mathbf{i} \cdot \mathbf{K} \\ \mathbf{J} \cdot \mathbf{i}' & \mathbf{J} \cdot \mathbf{J}' & \mathbf{J} \cdot \mathbf{K} \\ \mathbf{K} \cdot \mathbf{i}' & \mathbf{K} \cdot \mathbf{J}' & \mathbf{K} \cdot \mathbf{K} \end{bmatrix} \tag{A-5}$$

The elements of the latter matrix may be recognized as the direction cosines of the unit vectors of one coordinate system in the other system. One may then write

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [\gamma] \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} \tag{A-6}$$

With these identities, the kinetic energy from Eq. 2.6-9 may be put in matrix form. Beginning with

$$\dot{R}^2 = \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}' \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \tag{A-7}$$

and noting that $\int_V \rho dv = M$, the total mass, the first term becomes

$$\frac{1}{2}\dot{R}^2 \int_V \rho dv = \frac{1}{2} M \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}' \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \tag{A-8}$$

In the second term, the identity

$$(\Omega \times \mathbb{L})^2 = (\mathbb{L} \times \Omega) \cdot (\mathbb{L} \times \Omega) \tag{A-9}$$

may be written using Eqs. A-2,4 as

$$(\Omega\times\mathbb{L})^2 = \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}' \begin{bmatrix}0 & -z & y\\ z & 0 & -x\\ -y & x & 0\end{bmatrix}' \begin{bmatrix}0 & -z & y\\ z & 0 & -x\\ -y & x & 0\end{bmatrix} \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}$$

$$= \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}' \begin{bmatrix}(y^2+z^2) & -xy & -xz\\ -xy & (x^2+z^2) & -yz\\ -xz & -yz & (x^2+y^2)\end{bmatrix} \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix} \tag{A-10}$$

The coordinate system of the body is defined as the principal axis of inertia system, so that the off-diagonal elements integrate to zero, and the second term becomes

$$\frac{1}{2}\int_V (\Omega\times\mathbb{L})^2 \rho\, dV = \frac{1}{2} \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}' \begin{bmatrix}I_{xx} & & \\ & I_{yy} & \\ & & I_{zz}\end{bmatrix} \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix} \tag{A-11}$$

In the remaining terms of the kinetic energy, the displacement vector of the elemental volume occurs. If these terms are to be evaluated, some scheme must be devised by which the continuous displacements may be represented by the displacements of the discrete set of panel points. This is accomplished by a numerical procedure known as an interpolation scheme. Interpolation schemes relate the displacements in the regions between the panel points to the panel point displacements. Consider the next term in the kinetic energy. It may be written as

$$\frac{1}{2}\int_{V_i} \dot{\mathbb{P}}_i^2 \rho\, dV = \frac{1}{2}\int_{V_i} \begin{Bmatrix}\dot{P}_{x'}\\ \dot{P}_{y'}\\ \dot{P}_{z'}\end{Bmatrix}_i' \begin{Bmatrix}\dot{P}_{x'}\\ \dot{P}_{y'}\\ \dot{P}_{z'}\end{Bmatrix} \rho\, dV \tag{A-12}$$

since

$$\begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}_i' \begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix}_i = \begin{Bmatrix} P_{x'} \\ P_{y'} \\ P_{z'} \end{Bmatrix}_i' [\gamma]_i' [\gamma]_i \begin{Bmatrix} P_{x'} \\ P_{y'} \\ P_{z'} \end{Bmatrix}_i \tag{A-13}$$

and

$$[\gamma]_i' [\gamma]_i = \lceil 1 \rfloor \tag{A-14}$$

In terms of some interpolation scheme, each component of the continuous displacement (or velocity) is related to the values at the panel points. Thus,

$$\int_{V_i} \dot{P}_{xi}^2 \, \rho dv = \{\dot{P}_{x'}\}' [A_{x'x'}] \{\dot{P}_{x'}\} \tag{A-15}$$

The resulting form is simply a quadratic expansion in the panel point velocities. The values of the elements in the mass matrix $[A_{x'x'}]$ will depend on the interpolation scheme used, which in turn dictates the accuracy of the representation of the continuous system. The third term in the expansion of the kinetic energy is then

$$\frac{1}{2} \int_{Vi} \dot{\mathbb{P}}_i^2 \rho dV = \frac{1}{2} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}_i' \begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix}_i \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}_i \tag{A-16}$$

In the remaining terms, the subscript i is omitted until the final form for the kinetic energy is written, it being understood that all elements must be written in the proper coordinate system.

In the fourth term,

$$(\Omega \times \mathbb{P})^2 = \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' \begin{bmatrix} (P_y^2 + P_z^2) & -P_x P_y & -P_x P_z \\ -P_x P_y & (P_x^2 + P_z^2) & -P_y P_z \\ -P_x P_z & -P_y P_z & (P_x^2 + P_y^2) \end{bmatrix} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} \tag{A-17}$$

From the properties of a similarity transformation (see Ref. 1), this may be written

$$(\Omega\times \mathbb{P})^2 = \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}(P_{y'}^2+P_{z'}^2) & -P_{x'}P_{y'} & -P_{x'}P_{z'}\\ -P_{x'}P_{y'} & (P_{x'}^2+P_{z'}^2) & -P_{y'}P_{z'}\\ -P_{x'}P_{z'} & -P_{y'}P_{z'} & (P_{x'}^2+P_{y'}^2)\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix} \tag{A-18}$$

This may be verified by direct expansion. Finally,

$$\frac{1}{2}\int_{vi}(\Omega\times \mathbb{P})^2\rho\, dv =$$

$$\frac{1}{2}\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}(\{P_{y'}\}'[A_{y'y'}]\{P_{y'}\}+\{P_{z'}\}'[A_{z'z'}]\{P_{z'}\}) & -\{P_{x'}\}'[A_{x'y'}]\{P_{y'}\} & -\{P_{x'}\}'[A_{x'z'}]\{P_{z'}\}\\ -\{P_{x'}\}'[A_{x'y'}]\{P_{y'}\} & (\{P_{x'}\}'[A_{x'x'}]\{P_{x'}\}+\{P_{z'}\}'[A_{z'z'}]\{P_{z'}\}) & -\{P_{y'}\}'[A_{y'z'}]\{P_{z'}\}\\ -\{P_{x'}\}'[A_{x'z'}]\{P_{z'}\} & -\{P_{y'}\}'[A_{y'z'}]\{P_{z'}\} & (\{P_{x'}\}'[A_{x'x'}]\{P_{x'}\}+\{P_{y'}\}'[A_{y'y'}]\{P_{y'}\})\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix} \tag{A-19}$$

Thus, since the quadratic form expressed in Eq. A-18 has coupling terms between the components of the displacement, the form in Eq. A-19 will have mass matrices with the subscripts indicating the coupling. These mass matrices will differ from those on the main diagonal in general, due to the variation in mass density, panel point spacings, and possible variations in the interpolation schemes used in the various directions in the component coordinate system.

The fifth term may be written in either of two forms:

$$\dot{\mathbb{R}}\cdot\Omega\times\mathbb{P} = -\Omega\cdot\dot{\mathbb{R}}\times\mathbb{P} \tag{A-20}$$

Defining

$$[\Omega] = \begin{bmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{bmatrix} \tag{A-21}$$

$$[\mathcal{V}] = \begin{bmatrix} 0 & \mathcal{V}_z & -\mathcal{V}_y \\ -\mathcal{V}_z & 0 & \mathcal{V}_x \\ \mathcal{V}_y & -\mathcal{V}_x & 0 \end{bmatrix} \tag{A-22}$$

then

$$\dot{\mathbb{R}}\cdot\boldsymbol{\Omega}\times\mathbb{P} = -\begin{Bmatrix} \mathcal{V}_x \\ \mathcal{V}_y \\ \mathcal{V}_z \end{Bmatrix}'[\Omega]\begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix} = \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}'[\mathcal{V}]\begin{Bmatrix} P_x \\ P_y \\ P_z \end{Bmatrix} \tag{A-23}$$

and $\dot{\mathbb{R}}\cdot\int_{vi}\boldsymbol{\Omega}\times\mathbb{P}\rho\,dv =$

$$\left\{\begin{aligned} &-\begin{Bmatrix} \mathcal{V}_x \\ \mathcal{V}_y \\ \mathcal{V}_z \end{Bmatrix}'[\Omega][\gamma]\begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}'\begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix}\begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix} \\ &\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}'[\mathcal{V}][\gamma]\begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}'\begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix}\begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix} \end{aligned}\right. \tag{A-24}$$

The new type of mass matrix introduced here arises from forms of the type

$$\int_{vi} P_{x'}\rho\,dV = \{1\}'[A_{x'}]\{P_{x'}\} \tag{A-25}$$

The same interpolation scheme as in Eq. A-15 would be used to obtain this result, but the elements of the single subscript type of mass matrix will in general be different from those of the double subscript type. The following shorthand notation has been adapted:

$$
\begin{bmatrix} \{1\} & \{0\} & \{0\} \\ \{0\} & \{1\} & \{0\} \\ \{0\} & \{0\} & \{1\} \end{bmatrix} = \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix} \tag{A-26}
$$

The sizes of the indicated column vectors are dictated by the number of panel points whose displacements are indicated in the adjacent matrices.

The sixth term is

$$
\dot{\mathbb{R}} \cdot \int_{vi} \dot{\mathbb{P}} \rho dv = \begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \tag{A-27}
$$

In the seventh term,

$$
\Omega \cdot \mathbb{L} \times \dot{\mathbb{P}} = \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [\gamma] \begin{bmatrix} 0 & -z' & y' \\ z' & 0 & -x' \\ -y' & x' & 0 \end{bmatrix} \begin{Bmatrix} \dot{P}_{x'} \\ \dot{P}_{y'} \\ \dot{P}_{z'} \end{Bmatrix} \tag{A-28}
$$

and

$$
\Omega \cdot \int_{vi} \mathbb{L} \times \dot{\mathbb{P}} \rho dV =
$$

$$
\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{0\}' & -\{1\}'[z'A_{y'}] & \{1\}'[y'A_{z'}] \\ \{1\}'[z'A_{x'}] & \{0\}' & -\{1\}'[x'A_{z'}] \\ -\{1\}'[y'A_{x'}] & \{1\}'[x'A_{y'}] & \{0\}' \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \tag{A-29}
$$

The mass matrices containing components of the position vector $\mathbb{L}$ inside the matrix brackets to the left of the mass matrix symbol arise from integrations of the form

$$
\int_{vi} z' \dot{P}_{x'} \rho dv = \{1\}' [z'A_{x'}] \{\dot{P}_{x'}\} \tag{A-30}
$$

In the limit as the number of panel points becomes large, and the interpolation scheme becomes increasingly accurate,

$$\{1\}'[z'A_{x'}]\{\dot{P}_{x'}\} \longrightarrow \{z'\}'[A_{x'}]\{\dot{P}_{x'}\} \tag{A-31}$$

where the column matrix $\{z'\}$ indicates the positions of the panel points in the component coordinate system, along the z'-axis. Although the original forms will be retained in defining the kinetic energy, the equations of motion will be derived using the approximation that Eq. A-31 is true without reservations.

Likewise the eighth term is

$$\boldsymbol{\Omega}\cdot\int_{vi}\mathbb{P}\times\dot{\mathbb{P}}\rho\,dv =$$

$$\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}\{0\}' & -\{P_{z'}\}'[A_{z'y'}] & \{P_{y'}\}'[A_{y'z'}]\\ \{P_{z'}\}'[A_{z'x'}] & \{0\}' & -\{P_{x'}\}'[A_{x'z'}]\\ -\{P_{y'}\}'[A_{y'x'}] & \{P_{x'}\}'[A_{x'y'}] & \{0\}'\end{bmatrix}\begin{Bmatrix}\{\dot{P}_{x'}\}\\ \{\dot{P}_{y'}\}\\ \{\dot{P}_{z'}\}\end{Bmatrix} \tag{A-32}$$

In the ninth term,

$$(\boldsymbol{\Omega}\times\mathbb{L})\cdot(\boldsymbol{\Omega}\times\mathbb{P}) = (\mathbb{L}\times\mathbb{P})\cdot(\mathbb{P}\times\boldsymbol{\Omega})$$

$$= \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}0 & -z' & y'\\ z' & 0 & -x'\\ -y' & x' & 0\end{bmatrix}'\begin{bmatrix}0 & -P_{z'} & P_{y'}\\ P_{z'} & 0 & -P_{x'}\\ -P_{y'} & P_{x'} & 0\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix} \tag{A-33}$$

so that

$$\int (\Omega \times \mathbb{L}) \cdot (\Omega \times \mathbb{P})\, \rho\, dv =$$

$$\begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \left(\{1\}'[z'A_{z'}]\{P_{z'}\} + \{1\}'[y'A_{y'}]\{P_{y'}\}\right) & -\{1\}'[y'A_{x'}]\{P_{x'}\} & -\{1\}'[z'A_{x'}]\{P_{x'}\} \\ -\{1\}'[x'A_{y'}]\{P_{y'}\} & \left(\{1\}'[x'A_{x'}]\{P_{x}\} + \{1\}'[z'A_{z'}]\{P_{z'}\}\right) & -\{1\}[z'A_{y'}]\{P_{y'}\} \\ -\{1\}'[x'A_{z'}]\{P_{z}\} & -\{1\}[y'A_{z'}]\{P_{z'}\} & \left(\{1\}'[x'A_{x'}]\{P_{x}\} + \{1\}'[y'A_{y'}]\{P_{y'}\}\right) \end{bmatrix} [\gamma]' \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} \tag{A-34}$$

Each of these terms is written for i -th component and the sum of these is the kinetic energy. These terms are collected next. The summation on i goes from $i = 1$ to $i = N$. Instead of distributing subscripts throughout, the entire set of terms to be summed over is indexed. The index will fall on all elements except the rigid body velocities.

The desired form for the kinetic energy is written in Eq. A-35. It will be substituted into the modified set of Lagrange's equations, which are derived in Appendix B.

$$
\begin{aligned}
T = {} & \frac{1}{2} M \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}' \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} \\
& + \sum_{i=1}^{i=N} \left[\frac{1}{2} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}' \begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \right. \\
& + \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \\
& + \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [v][\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \\
& + \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{0\}' & -\{P_{z'}\}'[A_{z'y'}] & \{P_{y'}\}'[A_{y'z'}] \\ \{P_{z'}\}'[A_{z'x'}] & \{0\}' & -\{P_{x'}\}'[A_{x'z'}] \\ -\{P_{y'}\}'[A_{y'x'}] & \{P_{x'}\}'[A_{x'y'}] & \{0\}' \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix} \\
& + \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix}' [\gamma] \begin{bmatrix} \{0\}' & -\{1\}'[z'A_{y'}] & \{1\}'[y'A_{z'}] \\ \{1\}'[z'A_{x'}] & \{0\}' & -\{1\}'[x'A_{z'}] \\ -\{1\}'[yA_{x'}] & \{1\}'[x'A_{y'}] & \{0\}' \end{bmatrix} \begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}
\end{aligned}
\qquad \text{A-35)}
$$

$$
+\frac{1}{2}\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}(\{P_{y'}\}'[A_{y'y'}]\{P_{y'}\}+\{P_{z'}\}'[A_{z'z'}]\{P_{z'}\}) & -\{P_{x'}\}'[A_{x'y'}]\{P_{y'}\} & -\{P_{x'}\}'[A_{x'z'}]\{P_{z'}\}\\ -\{P_{y'}\}'[A_{y'x'}]\{P_{x'}\} & (\{P_{x'}\}'[A_{x'x'}]\{P_{x'}\}+\{P_{z'}\}'[A_{z'z'}]\{P_{z'}\}) & -\{P_{y'}\}'[A_{y'z'}]\{P_{z'}\}\\ -\{P_{z'}\}'[A_{x'z'}]\{P_{x'}\} & -\{P_{z'}\}'[A_{z'y'}]\{P_{y'}\} & (\{P_{x'}\}'[A_{x'x'}]\{P_{x'}\}+\{P_{y'}\}'[A_{y'y'}]\{P_{y'}\})\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}
$$

$$
+\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}'[\gamma]\begin{bmatrix}(\{1\}'[z'A_{z'}]\{P_{z'}\}+\{1\}'[y'A_{y'}]\{P_{y'}\}) & -\{1\}'[y'A_{x'}]\{P_{x'}\} & -\{1\}'[z'A_{x'}]\{P_{x'}\}\\ -\{1\}'[x'A_{y'}]\{P_{y'}\} & (\{1\}'[x'A_{x'}]\{P_{x'}\}+\{1\}'[z'A_{z'}]\{P_{z'}\}) & -\{1\}'[z'A_{y'}]\{P_{y'}\}\\ -\{1\}'[x'A_{z'}]\{P_{z'}\} & -\{1\}'[y'A_{z'}]\{P_{z'}\} & (\{1\}'[x'A_{x'}]\{P_{x'}\}+\{1\}'[y'A_{y'}]\{P_{y'}\})\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}\Bigg]_i
$$

End Summation

(A-35)

APPENDIX B

TRANSFORMATION TO THE MODIFIED FORM OF LAGRANGE'S EQUATIONS

The details of the derivation of the modified form of Lagrange's equations as presented in Section 2 are written here.

Lagrange's equations are first written as

$$\frac{d}{dt}\begin{Bmatrix} \dfrac{\partial L}{\partial \dot{X}} \\ \dfrac{\partial L}{\partial \dot{Y}} \\ \dfrac{\partial L}{\partial \dot{Z}} \end{Bmatrix} - \begin{Bmatrix} \dfrac{\partial L}{\partial X} \\ \dfrac{\partial L}{\partial Y} \\ \dfrac{\partial L}{\partial Z} \end{Bmatrix} = \begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} + \sum_j \sigma_j \begin{Bmatrix} \dfrac{\partial F_j}{\partial X} \\ \dfrac{\partial F_j}{\partial Y} \\ \dfrac{\partial F_j}{\partial Z} \end{Bmatrix} \tag{B-1}$$

$$\frac{d}{dt}\begin{Bmatrix} \dfrac{\partial L}{\partial \dot{\psi}} \\ \dfrac{\partial L}{\partial \dot{\theta}} \\ \dfrac{\partial L}{\partial \dot{\phi}} \end{Bmatrix} - \begin{Bmatrix} \dfrac{\partial L}{\partial \psi} \\ \dfrac{\partial L}{\partial \theta} \\ \dfrac{\partial L}{\partial \phi} \end{Bmatrix} = \begin{Bmatrix} N_\psi \\ N_\theta \\ N_\phi \end{Bmatrix} + \sum_j \sigma_j \begin{Bmatrix} \dfrac{\partial F_j}{\partial \psi} \\ \dfrac{\partial F_j}{\partial \theta} \\ \dfrac{\partial F_j}{\partial \phi} \end{Bmatrix} \tag{B-2}$$

$$\left(\frac{d}{dt}\begin{Bmatrix} \left\{\dfrac{\partial L}{\partial \dot{\rho}_{x'}}\right\} \\ \left\{\dfrac{\partial L}{\partial \dot{\rho}_{y'}}\right\} \\ \left\{\dfrac{\partial L}{\partial \dot{\rho}_{z'}}\right\} \end{Bmatrix} - \begin{Bmatrix} \left\{\dfrac{\partial L}{\partial \rho_{x'}}\right\} \\ \left\{\dfrac{\partial L}{\partial \rho_{y'}}\right\} \\ \left\{\dfrac{\partial L}{\partial \rho_{z'}}\right\} \end{Bmatrix} = \begin{Bmatrix} \{Q_{x'}\} \\ \{Q_{y'}\} \\ \{Q_{z'}\} \end{Bmatrix} + \sum_j \sigma_j \begin{Bmatrix} \left\{\dfrac{\partial F_j}{\partial \rho_{x'}}\right\} \\ \left\{\dfrac{\partial F_j}{\partial \rho_{y'}}\right\} \\ \left\{\dfrac{\partial F_j}{\partial \rho_{z'}}\right\} \end{Bmatrix}\right)_i \tag{B-3}$$

$$i = 1, 2, \cdots, N$$

where (X, Y, Z), (ψ, θ, ϕ), $(\{\rho_{x'}\}, \{\rho_{y'}\}, \{\rho_{z'}\})$ and their time derivatives completely specify the position and velocity of every particle in the system relative

to the inertial frame of reference. The Lagrangian, $L = T - U$, is expressed in terms of the body linear and angular velocities, (v_x, v_y, v_z) and $(\Omega_x, \Omega_y, \Omega_z)$, and the panel point displacements and velocities. These are not a suitable set of variables for use in the ordinary form of Lagrange's equations, as they do not specify the motion relative to an inertial frame of reference. The Eqs. B-1, 2,3 are valid, however, and may be transformed so that all operations in the equations act on the variables used to define the Lagrangian.

The transformation matrices $[\Gamma]$ and $[R]$ and the matrices $[\Omega]$ and $[v]$ may be recalled from Eqs. 2.7-4, 5, 11, 12. Their use as a brief notation is valuable here.

The first operation indicated in Eq. B-1 may be transformed by use of the chain rule for differentiation in calculus and the definition of the matrix $[\Gamma]$;

$$\begin{Bmatrix} \frac{\partial}{\partial \dot{X}} \\ \frac{\partial}{\partial \dot{Y}} \\ \frac{\partial}{\partial \dot{Z}} \end{Bmatrix} = \begin{bmatrix} \frac{\partial v_x}{\partial \dot{X}} & \frac{\partial v_y}{\partial \dot{X}} & \frac{\partial v_z}{\partial \dot{X}} \\ \frac{\partial v_x}{\partial \dot{Y}} & \frac{\partial v_y}{\partial \dot{Y}} & \frac{\partial v_z}{\partial \dot{Y}} \\ \frac{\partial v_x}{\partial \dot{Z}} & \frac{\partial v_y}{\partial \dot{Z}} & \frac{\partial v_z}{\partial \dot{Z}} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial v_x} \\ \frac{\partial}{\partial v_y} \\ \frac{\partial}{\partial v_z} \end{Bmatrix} = [\Gamma]' \begin{Bmatrix} \frac{\partial}{\partial v_x} \\ \frac{\partial}{\partial v_y} \\ \frac{\partial}{\partial v_z} \end{Bmatrix} \qquad \text{(B-4)}$$

so that

$$\begin{Bmatrix} \frac{\partial T}{\partial \dot{X}} \\ \frac{\partial T}{\partial \dot{Y}} \\ \frac{\partial T}{\partial \dot{Z}} \end{Bmatrix} = [\Gamma]' \begin{Bmatrix} \frac{\partial T}{\partial v_x} \\ \frac{\partial T}{\partial v_y} \\ \frac{\partial T}{\partial v_z} \end{Bmatrix} \qquad \text{(B-5)}$$

Since the kinetic energy is independent of the inertial displacements (X, Y, Z), then

$$\begin{Bmatrix} \frac{\partial T}{\partial X} \\ \frac{\partial T}{\partial Y} \\ \frac{\partial T}{\partial Z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{B-6}$$

If no kinematic constraints are imposed on the "rigid-body" motion of the vehicle, then (X, Y, Z) are independent coordinates, and

$$\begin{Bmatrix} \frac{\partial F_j'}{\partial X} \\ \frac{\partial F_j'}{\partial Y} \\ \frac{\partial F_j'}{\partial Z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{B-7}$$

The time derivative of Eq. B-5 is

$$\frac{d}{dt} \begin{Bmatrix} \frac{\partial T}{\partial \dot{X}} \\ \frac{\partial T}{\partial \dot{Y}} \\ \frac{\partial T}{\partial \dot{Z}} \end{Bmatrix} = \left[\frac{d\Gamma}{dt}\right]' \begin{Bmatrix} \frac{\partial T}{\partial v_x} \\ \frac{\partial T}{\partial v_y} \\ \frac{\partial T}{\partial v_z} \end{Bmatrix} + \left[\Gamma\right]' \frac{d}{dt} \begin{Bmatrix} \frac{\partial T}{\partial v_x} \\ \frac{\partial T}{\partial v_y} \\ \frac{\partial T}{\partial v_z} \end{Bmatrix} \tag{B-8}$$

Since the potential energy is not dependent on the body coordinates, then Eq. B-1 is written using Eqs. B-6, 7, 8 as

$$[\Gamma]' \frac{d}{dt} \begin{Bmatrix} \dfrac{\partial T}{\partial v_x} \\ \dfrac{\partial T}{\partial v_y} \\ \dfrac{\partial T}{\partial v_z} \end{Bmatrix} + \left[\frac{d\Gamma}{dt}\right]' \begin{Bmatrix} \dfrac{\partial T}{\partial v_x} \\ \dfrac{\partial T}{\partial v_y} \\ \dfrac{\partial T}{\partial v_z} \end{Bmatrix} = \begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} \tag{B-9}$$

The generalized forces are transformed by

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix} = [\Gamma] \begin{Bmatrix} Q_X \\ Q_Y \\ Q_Z \end{Bmatrix} \tag{B-10}$$

into the body coordinate system. Premultiplication by $[\Gamma]$ of Eq. B-9 together with the identity $[\Gamma][\Gamma]' = [1]$ and

$$[\Gamma]\left[\frac{d\Gamma}{dt}\right]' = -[\Omega] = -\begin{bmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{bmatrix} \tag{B-11}$$

yields one of the desired transformed equations

$$\left([I]\frac{d}{dt}-[\Omega]\right)\begin{Bmatrix}\dfrac{\partial T}{\partial v_x}\\[2mm]\dfrac{\partial T}{\partial v_y}\\[2mm]\dfrac{\partial T}{\partial v_z}\end{Bmatrix}=\begin{Bmatrix}Q_x\\[2mm]Q_y\\[2mm]Q_z\end{Bmatrix} \tag{B-12}$$

The transformation of Eq. B-2 is a bit more tedious. Using the chain rule of differentiation and the definition of $[R]$,

$$\begin{Bmatrix}\dfrac{\partial T}{\partial \dot\psi}\\[2mm]\dfrac{\partial T}{\partial \dot\theta}\\[2mm]\dfrac{\partial T}{\partial \dot\phi}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial \Omega_x}{\partial \dot\psi} & \dfrac{\partial \Omega_y}{\partial \dot\psi} & \dfrac{\partial \Omega_z}{\partial \dot\psi}\\[2mm]\dfrac{\partial \Omega_x}{\partial \dot\theta} & \dfrac{\partial \Omega_y}{\partial \dot\theta} & \dfrac{\partial \Omega_z}{\partial \dot\theta}\\[2mm]\dfrac{\partial \Omega_x}{\partial \dot\phi} & \dfrac{\partial \Omega_y}{\partial \dot\phi} & \dfrac{\partial \Omega_z}{\partial \dot\phi}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial T}{\partial \Omega_x}\\[2mm]\dfrac{\partial T}{\partial \Omega_y}\\[2mm]\dfrac{\partial T}{\partial \Omega_z}\end{Bmatrix}=[R]'\begin{Bmatrix}\dfrac{\partial T}{\partial \Omega_x}\\[2mm]\dfrac{\partial T}{\partial \Omega_y}\\[2mm]\dfrac{\partial T}{\partial \Omega_z}\end{Bmatrix} \tag{B-13}$$

also, the chain rule yields

$$\begin{Bmatrix}\dfrac{\partial T}{\partial \psi}\\[2mm]\dfrac{\partial T}{\partial \theta}\\[2mm]\dfrac{\partial T}{\partial \phi}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial v_x}{\partial \psi} & \dfrac{\partial v_y}{\partial \psi} & \dfrac{\partial v_z}{\partial \psi}\\[2mm]\dfrac{\partial v_x}{\partial \theta} & \dfrac{\partial v_y}{\partial \theta} & \dfrac{\partial v_z}{\partial \theta}\\[2mm]\dfrac{\partial v_x}{\partial \phi} & \dfrac{\partial v_y}{\partial \phi} & \dfrac{\partial v_z}{\partial \phi}\end{Bmatrix}\begin{Bmatrix}\dfrac{\partial T}{\partial v_x}\\[2mm]\dfrac{\partial T}{\partial v_y}\\[2mm]\dfrac{\partial T}{\partial v_z}\end{Bmatrix}+\begin{bmatrix}\dfrac{\partial \Omega_x}{\partial \psi} & \dfrac{\partial \Omega_y}{\partial \psi} & \dfrac{\partial \Omega_z}{\partial \psi}\\[2mm]\dfrac{\partial \Omega_x}{\partial \theta} & \dfrac{\partial \Omega_y}{\partial \theta} & \dfrac{\partial \Omega_z}{\partial \theta}\\[2mm]\dfrac{\partial \Omega_x}{\partial \phi} & \dfrac{\partial \Omega_y}{\partial \phi} & \dfrac{\partial \Omega_z}{\partial \phi}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial T}{\partial \Omega_x}\\[2mm]\dfrac{\partial T}{\partial \Omega_y}\\[2mm]\dfrac{\partial T}{\partial \Omega_z}\end{Bmatrix} \tag{B-14}$$

which may be shown to produce

$$\begin{Bmatrix} \frac{\partial T}{\partial \psi} \\ \frac{\partial T}{\partial \theta} \\ \frac{\partial T}{\partial \phi} \end{Bmatrix} = [R]'[\mathcal{N}] \begin{Bmatrix} \frac{\partial T}{\partial \mathcal{N}_x} \\ \frac{\partial T}{\partial \mathcal{N}_y} \\ \frac{\partial T}{\partial \mathcal{N}_z} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\dot{\psi}\cos\theta & -\dot{\psi}\sin\theta\sin\phi & -\dot{\psi}\sin\theta\cos\phi \\ 0 & \dot{\psi}\cos\phi\cos\theta-\dot{\theta}\sin\phi & -\dot{\psi}\sin\phi\cos\theta+\dot{\theta}\cos\phi \end{bmatrix} \begin{Bmatrix} \frac{\partial T}{\partial \Omega_x} \\ \frac{\partial T}{\partial \Omega_y} \\ \frac{\partial T}{\partial \Omega_z} \end{Bmatrix} \tag{B-15}$$

using the definition of $[\mathcal{N}]$.

Differentiation of Eq. B-13 with respect to time yields

$$\frac{d}{dt}\begin{Bmatrix} \frac{\partial T}{\partial \dot{\psi}} \\ \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{Bmatrix} = [R]'\frac{d}{dt} \begin{Bmatrix} \frac{\partial T}{\partial \Omega_x} \\ \frac{\partial T}{\partial \Omega_y} \\ \frac{\partial T}{\partial \Omega_z} \end{Bmatrix} + \begin{bmatrix} -\dot{\theta}\cos\theta & 0 & 0 \\ \dot{\phi}\cos\phi\cos\theta-\dot{\theta}\sin\phi\sin\theta & -\dot{\phi}\sin\phi & 0 \\ -\dot{\phi}\sin\phi\cos\theta-\dot{\theta}\cos\phi\sin\theta & -\dot{\phi}\cos\phi & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial T}{\partial \Omega_x} \\ \frac{\partial T}{\partial \Omega_y} \\ \frac{\partial T}{\partial \Omega_z} \end{Bmatrix} \tag{B-16}$$

If no kinematic constraints are imposed on the "rigid-body" motion of the vehicle, then (ψ, θ, ϕ) are independent coordinates and

$$\begin{Bmatrix} \frac{\partial F_j}{\partial \psi} \\ \frac{\partial F_j}{\partial \theta} \\ \frac{\partial F_j}{\partial \phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{B-17}$$

Substitution of Eqs. B-15, 16, 17 into Eq. B-2 yields

$$\left([R]'\frac{d}{dt} - [R]'[\Omega]\right)\begin{Bmatrix} \frac{\partial T}{\partial \Omega_x} \\ \frac{\partial T}{\partial \Omega_y} \\ \frac{\partial T}{\partial \Omega_z} \end{Bmatrix} - [R]'[\nu]\begin{Bmatrix} \frac{\partial T}{\partial \nu_x} \\ \frac{\partial T}{\partial \nu_y} \\ \frac{\partial T}{\partial \nu_z} \end{Bmatrix} = \begin{Bmatrix} N_\psi \\ N_\theta \\ N_\phi \end{Bmatrix} \qquad \text{(B-18)}$$

since the terms explicitly written in Eq. B-15 and Eq. B-16, when subtracted, yield - $[R]'[\nu]$

The components $(N_\psi, N_\theta, N_\phi)$ of Eq. B-18 are the generalized forces associated with the coordinates (ψ, θ, ϕ). They are, physically, moments about the line of nodes, and are not mutually orthogonal components of a vector. They transform into the body coordinate system by

$$\begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix} = [R]\begin{Bmatrix} N_\psi \\ N_\theta \\ N_\phi \end{Bmatrix} \qquad \text{(B-19)}$$

Premultiplication of Eq. B-18 by $[R]$ then yields the modified form

$$\left([\,1\,]\frac{d}{dt} - [\Omega]\right)\begin{Bmatrix} \frac{\partial T}{\partial \Omega_x} \\ \frac{\partial T}{\partial \Omega_y} \\ \frac{\partial T}{\partial \Omega_z} \end{Bmatrix} - \begin{Bmatrix} \frac{\partial T}{\partial \nu_x} \\ \frac{\partial T}{\partial \nu_y} \\ \frac{\partial T}{\partial \nu_z} \end{Bmatrix}[\nu] = \begin{Bmatrix} N_x \\ N_y \\ N_z \end{Bmatrix} \qquad \text{(B-20)}$$

The desired transformations are thus obtained. Lagrange's equations in modified form are

$$\left(\left[1\right]\frac{d}{dt}-\left[\Omega\right]\right)\begin{Bmatrix}\dfrac{\partial T}{\partial v_x}\\ \dfrac{\partial T}{\partial v_y}\\ \dfrac{\partial T}{\partial v_z}\end{Bmatrix}=\begin{Bmatrix}Q_x\\ Q_y\\ Q_z\end{Bmatrix} \tag{B-21}$$

$$\left(\left[1\right]\frac{d}{dt}-\left[\Omega\right]\right)\begin{Bmatrix}\dfrac{\partial T}{\partial \Omega_x}\\ \dfrac{\partial T}{\partial \Omega_y}\\ \dfrac{\partial T}{\partial \Omega_z}\end{Bmatrix}-\left[v\right]\begin{Bmatrix}\dfrac{\partial T}{\partial v_x}\\ \dfrac{\partial T}{\partial v_y}\\ \dfrac{\partial T}{\partial v_z}\end{Bmatrix}=\begin{Bmatrix}N_x\\ N_y\\ N_z\end{Bmatrix} \tag{B-22}$$

$$\left(\frac{d}{dt}\begin{Bmatrix}\left\{\dfrac{\partial L}{\partial \dot P_{x'}}\right\}\\ \left\{\dfrac{\partial L}{\partial \dot P_{y'}}\right\}\\ \left\{\dfrac{\partial L}{\partial \dot P_{z'}}\right\}\end{Bmatrix}-\begin{Bmatrix}\left\{\dfrac{\partial L}{\partial P_{x'}}\right\}\\ \left\{\dfrac{\partial L}{\partial P_{y'}}\right\}\\ \left\{\dfrac{\partial L}{\partial P_{z'}}\right\}\end{Bmatrix}-\begin{Bmatrix}\left\{Q_{x'}\right\}\\ \left\{Q_{y'}\right\}\\ \left\{Q_{z'}\right\}\end{Bmatrix}+\sum_j \sigma_j\begin{Bmatrix}\left\{\dfrac{\partial F_j}{\partial P_{x'}}\right\}\\ \left\{\dfrac{\partial F_j}{\partial P_{y'}}\right\}\\ \left\{\dfrac{\partial F_j}{\partial P_{z'}}\right\}\end{Bmatrix}\right)_i \tag{B-23}$$

$$i=1,2,\cdots,N$$

APPENDIX C

DERIVATION OF THE EQUATIONS OF MOTION

Some of the details in the derivation of the equations of motion will be presented here.

The forms for the kinetic and potential energy of the system will be substituted into the modified set of Lagrange's equations. A number of useful identities will first be established in order that the operations indicated in the modified equations may be performed in matrix form rather than individually.

If an arbitrary function, G_1, has the form

$$G_1 = \begin{Bmatrix} K_x \\ K_y \\ K_z \end{Bmatrix}' \begin{Bmatrix} L_x \\ L_y \\ K_z \end{Bmatrix} \tag{C-1}$$

where the indicated variables are also arbitrary, then it is easily shown that

$$\begin{Bmatrix} \dfrac{\partial G_1}{\partial K_x} \\ \dfrac{\partial G_1}{\partial K_y} \\ \dfrac{\partial G_1}{\partial K_z} \end{Bmatrix} = \begin{Bmatrix} L_x \\ L_y \\ L_z \end{Bmatrix} \tag{C-2}$$

and

$$\begin{Bmatrix} \dfrac{\partial G_1}{\partial L_x} \\ \dfrac{\partial G_1}{\partial L_y} \\ \dfrac{\partial G_1}{\partial L_z} \end{Bmatrix} = \begin{Bmatrix} K_x \\ K_y \\ K_z \end{Bmatrix} \tag{C-3}$$

also, in the same sense, if

$$G_2 = \begin{Bmatrix} K_x \\ K_y \\ K_z \end{Bmatrix}' \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_z \end{Bmatrix} \tag{C-4}$$

then

$$\begin{Bmatrix} \dfrac{\partial G_2}{\partial K_x} \\[2ex] \dfrac{\partial G_2}{\partial K_y} \\[2ex] \dfrac{\partial G_2}{\partial K_z} \end{Bmatrix} = \left([C] + [C]' \right) \begin{Bmatrix} K_x \\[2ex] K_y \\[2ex] K_z \end{Bmatrix} \tag{C-5}$$

With the use of these identities and the alternate form of Eq. A-24, the following operation on the kinetic energy is obvious:

$$\begin{Bmatrix} \dfrac{\partial T}{\partial v_x} \\[2ex] \dfrac{\partial T}{\partial v_y} \\[2ex] \dfrac{\partial T}{\partial v_z} \end{Bmatrix} = M \begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} + \sum_{i=1}^{i=N} \left([\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{\rho}_{x'}\} \\ \{\dot{\rho}_{y'}\} \\ \{\dot{\rho}_{z'}\} \end{Bmatrix} \right.$$

$$\left. - [\Omega]\,[\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\rho_{x'}\} \\ \{\rho_{y'}\} \\ \{\rho_{z'}\} \end{Bmatrix} \right)_i \tag{C-6}$$

The equations of motion governing the body linear velocities follow immediately from Eq. B-21, and are

$$
\begin{aligned}
& M\begin{Bmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{Bmatrix} - M[\Omega]\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} \\
& + \sum_{i=1}^{i=N}\left([\gamma]\begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\ddot{\rho}_{x'}\} \\ \{\ddot{\rho}_{y'}\} \\ \{\ddot{\rho}_{z'}\} \end{Bmatrix} \right. \\
& -2[\Omega][\gamma]\begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{\rho}_{x'}\} \\ \{\dot{\rho}_{y'}\} \\ \{\dot{\rho}_{z'}\} \end{Bmatrix} \qquad \text{(C-7)} \\
& \left. + \left([\Omega][\Omega]-[\dot{\Omega}]\right)[\gamma]\begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix}' \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\rho_{x'}\} \\ \{\rho_{y'}\} \\ \{\rho_{z'}\} \end{Bmatrix} \right)_i = \begin{Bmatrix} Q_x \\ Q_y \\ Q_z \end{Bmatrix}
\end{aligned}
$$

The derivation of the equations governing the angular velocities of the body are more lengthy due to the number of terms in the kinetic energy involving the angular velocities. Using the previous identities, the remaining operation in Eq. B-22 is written in Eq. C-8.

$$
\begin{aligned}
\begin{Bmatrix} \frac{\partial T}{\partial \Omega_x} \\ \frac{\partial T}{\partial \Omega_y} \\ \frac{\partial T}{\partial \Omega_z} \end{Bmatrix} = {} & \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} + \sum_{i=1}^{i=N} \Bigg[[N][\gamma] \begin{bmatrix} \{1\} & & \\ & \{1\} & \\ & & \{1\} \end{bmatrix} \begin{bmatrix} [A_{x'}] & & \\ & [A_{y'}] & \\ & & [A_{z'}] \end{bmatrix} \begin{Bmatrix} \{\dot{\rho}_{x'}\} \\ \{\dot{\rho}_{y'}\} \\ \{\dot{\rho}_{z'}\} \end{Bmatrix} \\
& + 2[\gamma] \begin{bmatrix} (\{\rho_{y'}\}'[A_{y'y'}]\{\rho_{y'}\} + \{\rho_{z'}\}'[A_{z'z'}]\{\rho_{z'}\}) & -\{\rho_{x'}\}'[A_{x'y'}]\{\rho_{y'}\} & -\{\rho_{x'}\}'[A_{x'z'}]\{\rho_{z'}\} \\ -\{\rho_{x'}\}'[A_{x'y'}]\{\rho_{y'}\} & (\{\rho_{x'}\}'[A_{x'x'}]\{\rho_{x'}\} + \{\rho_{z'}\}'[A_{z'z'}]\{\rho_{z'}\}) & -\{\rho_{y'}\}'[A_{y'z'}]\{\rho_{z'}\} \\ -\{\rho_{x'}\}'[A_{x'z'}]\{\rho_{z'}\} & -\{\rho_{y'}\}'[A_{y'z'}]\{\rho_{z'}\} & (\{\rho_{x'}\}'[A_{x'x'}]\{\rho_{x'}\} + \{\rho_{y'}\}'[A_{y'y'}]\{\rho_{y'}\}) \end{bmatrix} [\gamma]' \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} \\
& + [\gamma] \left(\begin{bmatrix} \{0\}' & -\{\rho_{z'}\}'[A_{z'y'}] & \{\rho_{y'}\}'[A_{y'z'}] \\ \{\rho_{z'}\}'[A_{z'x'}] & \{0\}' & -\{\rho_{x'}\}'[A_{x'z'}] \\ -\{\rho_{y'}\}'[A_{y'x'}] & \{\rho_{x'}\}'[A_{x'y'}] & \{0\}' \end{bmatrix} \begin{bmatrix} \{0\}' & -\{1\}'[z'A_{y'}] & \{1\}'[y'A_{z'}] \\ \{1\}'[z'A_{x'}] & \{0\}' & -\{1\}'[x'A_{z'}] \\ -\{1\}'[y'A_{x'}] & \{1\}'[x'A_{y'}] & \{0\}' \end{bmatrix} \right) \begin{Bmatrix} \{\dot{\rho}_{x'}\} \\ \{\dot{\rho}_{y'}\} \\ \{\dot{\rho}_{z'}\} \end{Bmatrix} \\
& + 2[\gamma] \begin{bmatrix} (\{1\}'[z'A_{z'}]\{\rho_{z'}\} + \{1\}'[y'A_{y'}]\{\rho_{y'}\}) & -\{1\}'[y'A_{x'}]\{\rho_{x'}\} & -\{1\}'[z'A_{x'}]\{\rho_{x'}\} \\ -\{1\}'[x'A_{y'}]\{\rho_{y'}\} & (\{1\}'[x'A_{x'}]\{\rho_{x'}\} + \{1\}'[z'A_{z'}]\{\rho_{z'}\}) & -\{1\}'[z'A_{y'}]\{\rho_{y'}\} \\ -\{1\}'[x'A_{z'}]\{\rho_{z'}\} & -\{1\}'[y'A_{z'}]\{\rho_{z'}\} & (\{1\}'[x'A_{x'}]\{\rho_{x'}\} + \{1\}'[y'A_{y'}]\{\rho_{y'}\}) \end{bmatrix} [\gamma]' \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} \Bigg]
\end{aligned}
\tag{C-8}
$$

This form together with that in Eq. C-6 is substituted into the second of the modified equations. A time-dependent moment of inertia is defined by collecting together all the resulting terms which multiply the angular accelerations. The moment of inertia matrix is given by Eq. C-9. In terms of this matrix, the equations governing the angular velocities of the body are given by Eq. C-10.

$$
\begin{aligned}
[I] = {} & \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} \\
& + \sum_{i=1}^{i=N} \Bigg([\gamma] \begin{bmatrix}
\left(\{\rho_{y'}\}'[A_{y'y'}]\{\rho_{y'}\} + \{\rho_{z'}\}'[A_{z'z'}]\{\rho_{z'}\}\right) & -\{\rho_{y'}\}'[A_{y'x'}]\{\rho_{x'}\} & -\{\rho_{z'}\}'[A_{z'x'}]\{\rho_{x'}\} \\
-\{\rho_{x'}\}'[A_{x'y'}]\{\rho_{y'}\} & \left(\{\rho_{x'}\}'[A_{x'x'}]\{\rho_{x'}\} + \{\rho_{z'}\}'[A_{z'z'}]\{\rho_{z'}\}\right) & -\{\rho_{z'}\}'[A_{z'y'}]\{\rho_{y'}\} \\
-\{\rho_{x'}\}'[A_{x'z'}]\{\rho_{z'}\} & -\{\rho_{y'}\}'[A_{y'z'}]\{\rho_{z'}\} & \left(\{\rho_{x'}\}'[A_{x'x'}]\{\rho_{x'}\} + \{\rho_{y'}\}'[A_{y'y'}]\{\rho_{y'}\}\right)
\end{bmatrix} [\gamma]' \\
& + 2[\gamma] \begin{bmatrix}
\left(\{1\}'[z'A_{z'}]\{\rho_{z'}\} + \{1\}'[y'A_{y'}]\{\rho_{y'}\}\right) & -\{1\}'[x'A_{y'}]\{\rho_{y'}\} & -\{1\}[x'A_{z'}]\{\rho_{z'}\} \\
-\{1\}'[y'A_{x'}]\{\rho_{x'}\} & \left(\{1\}'[x'A_{x'}]\{\rho_{x'}\} + \{1\}'[z'A_{z'}]\{\rho_{z'}\}\right) & -\{1\}'[y'A_{z'}]\{\rho_{z'}\} \\
-\{1\}'[z'A_{x'}]\{\rho_{x'}\} & -\{1\}'[z'A_{y'}]\{\rho_{y'}\} & \left(\{1\}'[x'A_{x'}]\{\rho_{x'}\} + \{1\}'[y'A_{y'}]\{\rho_{y'}\}\right)
\end{bmatrix} \Bigg)_i
\end{aligned}
\qquad \text{(C-9)}
$$

$$
\begin{aligned}
&[I]\begin{Bmatrix}\dot\Omega_x\\ \dot\Omega_y\\ \dot\Omega_z\end{Bmatrix}-[\Omega][I]\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}+\sum_{i=1}^{i=N}\Bigg[\left([\dot N]+[N][\Omega]-[\Omega][N]\right)[\gamma]\begin{bmatrix}\{1\} & & \\ & \{1\} & \\ & & \{1\}\end{bmatrix}'\begin{bmatrix}[A_x'] & & \\ & [A_y'] & \\ & & [A_z']\end{bmatrix}\begin{Bmatrix}\{\rho_x'\}\\ \{\rho_y'\}\\ \{\rho_z'\}\end{Bmatrix}\\
&-[\Omega][\gamma]\left(\begin{bmatrix}\{0\}' & -\{\rho_z'\}'[A_z'y'] & \{\rho_y'\}'[A_y'z']\\ \{\rho_z'\}'[A_z'x'] & \{0\} & -\{\rho_x'\}'[A_x'z']\\ \{\rho_y'\}'[A_y'x'] & \{\rho_x'\}'[A_x'y'] & \{0\}'\end{bmatrix}+\begin{bmatrix}\{0\}' & -\{1\}'[z'A_y'] & \{1\}'[y'A_z']\\ \{1\}'[z'A_x'] & \{0\} & -\{1\}'[x'A_z']\\ -\{1\}[y'A_x'] & \{1\}'[x'A_y'] & \{0\}\end{bmatrix}\right)\begin{Bmatrix}\{\dot\rho_x'\}\\ \{\dot\rho_y'\}\\ \{\dot\rho_z'\}\end{Bmatrix}\\
&+[\gamma]\left(\begin{bmatrix}\{0\}' & -\{\rho_z'\}'[A_z'y'] & \{\rho_y'\}'[A_y'z']\\ \{\rho_z'\}'[A_z'x'] & \{0\}' & -\{\rho_x'\}'[A_x'z']\\ -\{\rho_y'\}'[A_y'x'] & \{\rho_x'\}'[A_x'y'] & \{0\}'\end{bmatrix}+\begin{bmatrix}\{0\}' & -\{1\}'[z'A_y'] & \{1\}'[y'A_z']\\ \{1\}'[z'A_x'] & \{0\} & -\{1\}'[x'A_z']\\ -\{1\}'[y'A_x'] & \{1\}'[x'A_y'] & \{0\}'\end{bmatrix}\right)\begin{Bmatrix}\{\ddot\rho_x'\}\\ \{\ddot\rho_y'\}\\ \{\ddot\rho_z'\}\end{Bmatrix}\\
&+[\gamma]\begin{bmatrix}2(\{\rho_y'\}'[A_y'y']\{\dot\rho_y'\}+\{\rho_z'\}'[A_z'z']\{\dot\rho_z'\}) & -(\{\dot\rho_x'\}'[A_x'y']\{\rho_y'\}+\{\rho_x'\}'[A_x'y']\{\dot\rho_y'\}) & -(\{\dot\rho_x'\}'[A_x'z']\{\rho_z'\}+\{\rho_x'\}'[A_x'z']\{\dot\rho_z'\})\\ -(\{\dot\rho_y'\}'[A_y'x']\{\rho_x'\}+\{\rho_y'\}'[A_y'x']\{\dot\rho_x'\}) & 2(\{\rho_x'\}'[A_x'x']\{\dot\rho_x'\}+\{\rho_z'\}'[A_z'z']\{\dot\rho_z'\}) & -(\{\dot\rho_y'\}'[A_y'z']\{\rho_z'\}+\{\rho_y'\}'[A_y'z']\{\dot\rho_z'\})\\ -(\{\dot\rho_z'\}'[A_z'x']\{\rho_x'\}+\{\rho_z'\}'[A_z'x']\{\dot\rho_x'\}) & -(\{\dot\rho_z'\}'[A_z'y']\{\rho_y'\}+\{\rho_z'\}'[A_z'y']\{\dot\rho_y'\}) & 2(\{\rho_x'\}'[A_x'x']\{\dot\rho_x'\}+\{\rho_y'\}'[A_y'y']\{\dot\rho_y'\})\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}\\
&-2[\gamma]\begin{bmatrix}(\{1\}'[z'A_z']\{\dot\rho_z'\}+\{1\}'[y'A_y']\{\dot\rho_y'\}) & -\{1\}'[y'A_x']\{\dot\rho_x'\} & -\{1\}'[z'A_x']\{\dot\rho_x'\}\\ -\{1\}'[x'A_y']\{\dot\rho_y'\} & (\{1\}'[x'A_x']\{\dot\rho_x'\}+\{1\}'[z'A_z']\{\dot\rho_z'\}) & -\{1\}'[z'A_y']\{\dot\rho_y'\}\\ -\{1\}'[x'A_z']\{\dot\rho_z'\} & -\{1\}'[y'A_z']\{\dot\rho_z'\} & (\{1\}'[x'A_x']\{\dot\rho_x'\}+\{1\}'[y'A_y']\{\dot\rho_y'\})\end{bmatrix}[\gamma]'\begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix}\Bigg]=\begin{Bmatrix}N_x\\ N_y\\ N_z\end{Bmatrix}
\end{aligned}
\tag{C-10}
$$

The panel point equations are more easily derived if use is made of the identities

$$\begin{Bmatrix}\Omega_{x'}\\ \Omega_{y'}\\ \Omega_{z'}\end{Bmatrix}_i = [\gamma]_i' \begin{Bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{Bmatrix} \qquad \begin{Bmatrix}v_{x'}\\ v_{y'}\\ v_{z'}\end{Bmatrix}_i = [\gamma]_i' \begin{Bmatrix}v_x\\ v_y\\ v_z\end{Bmatrix} \tag{C-11}$$

The rigid body velocities are then written in each component coordinate system. It will be understood that primes on their subscripts mean that they must be written in the i-th system. This avoids considerable complication in the equations, for many of the terms cannot be broken down into simple product form.

The same identities for column operators may be used to advantage where panel point displacements appear in ordered column form. Where they appear in complicated matrix form, the matrix may be expanded and the operations performed separately. Noting that

$$\begin{bmatrix} \{0\}' & -\{\rho_{z'}\}'[A_{z'y'}] & \{\rho_{y'}\}'[A_{y'z'}] \\ \{\rho_{z'}\}[A_{z'x'}] & \{0\}' & -\{\rho_{x'}\}'[A_{x'z'}] \\ -\{\rho_{y'}\}'[A_{y'x'}] & \{\rho_{x'}\}'[A_{x'y'}] & \{0\}' \end{bmatrix}' [\gamma] \begin{Bmatrix}\dot{\Omega}_x\\ \Omega_y\\ \dot{\Omega}_z\end{Bmatrix} \tag{C-12}$$

$$= \begin{bmatrix} [0] & -\dot{\Omega}_{z'}[A_{x'y'}] & \dot{\Omega}_{y'}[A_{x'z'}] \\ \dot{\Omega}_{z'}[A_{y'x'}] & [0] & -\Omega_{x'}[A_{y'z'}] \\ -\dot{\Omega}_{y'}[A_{z'x'}] & \dot{\Omega}_{x'}[A_{z'y'}] & [0] \end{bmatrix} \begin{Bmatrix}\{\rho_{x'}\}\\ \{\rho_{y'}\}\\ \{\rho_{z'}\}\end{Bmatrix}$$

(note also the primes on the $\dot{\Omega}$ subscripts), the panel point equations may be written in the form of Eq. C-13.

$$
\begin{bmatrix} [A_{x'x'}] & & \\ & [A_{y'y'}] & \\ & & [A_{z'z'}] \end{bmatrix}
\begin{Bmatrix} \{\ddot{P}_{x'}\} \\ \{\ddot{P}_{y'}\} \\ \{\ddot{P}_{z'}\} \end{Bmatrix}
+
\begin{bmatrix} [\,0\,] & -2\Omega_{z'}[A_{x'y'}] & 2\Omega_{y'}[A_{x'z'}] \\ 2\Omega_{z'}[A_{y'x'}] & [\,0\,] & -2\Omega_{x'}[A_{y'z'}] \\ -2\Omega_{y'}[A_{z'x'}] & 2\Omega_{x'}[A_{z'y'}] & [\,0\,] \end{bmatrix}
\begin{Bmatrix} \{\dot{P}_{x'}\} \\ \{\dot{P}_{y'}\} \\ \{\dot{P}_{z'}\} \end{Bmatrix}
$$

$$
+
\begin{bmatrix} (\dot{v}_{x'}+\Omega_{y'}v_{z'}-\Omega_{z'}v_{y'})[A_{x'}]' & & \\ & (\dot{v}_{y'}+\Omega_{z'}v_{x'}-\Omega_{x'}v_{z'})[A_{y'}]' & \\ & & (\dot{v}_{z'}+\Omega_{x'}v_{y'}-\Omega_{y'}v_{x'})[A_{z'}]' \end{bmatrix}
\begin{Bmatrix} \{1\} \\ \{1\} \\ \{1\} \end{Bmatrix}
$$

$$
+
\begin{bmatrix} -(\Omega_{y'}^2+\Omega_{z'}^2)[A_{x'x'}] & (-\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{x'y'}] & (\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{x'z'}] \\ (\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{y'x'}] & -(\Omega_{x'}^2+\Omega_{z'}^2)[A_{y'y'}] & (-\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{y'z'}] \\ (-\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{z'x'}] & (\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{z'y'}] & -(\Omega_{x'}^2+\Omega_{y'}^2)[A_{z'z'}] \end{bmatrix}
\begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix}
$$

$$
+
\begin{bmatrix} -(\Omega_{y'}^2+\Omega_{z'}^2)[A_{x'}]' & (-\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{x'}]' & (\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{x'}]' \\ (\dot{\Omega}_{z'}+\Omega_{x'}\Omega_{y'})[A_{y'}]' & -(\Omega_{x'}^2+\Omega_{z'}^2)[A_{y'}]' & (-\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{y'}]' \\ (-\dot{\Omega}_{y'}+\Omega_{x'}\Omega_{z'})[A_{z'}]' & (\dot{\Omega}_{x'}+\Omega_{y'}\Omega_{z'})[A_{z'}]' & -(\Omega_{x'}^2+\Omega_{y'}^2)[A_{z'}]' \end{bmatrix}
\begin{Bmatrix} \{x'\} \\ \{y'\} \\ \{z'\} \end{Bmatrix}
\qquad \text{(C-13)}
$$

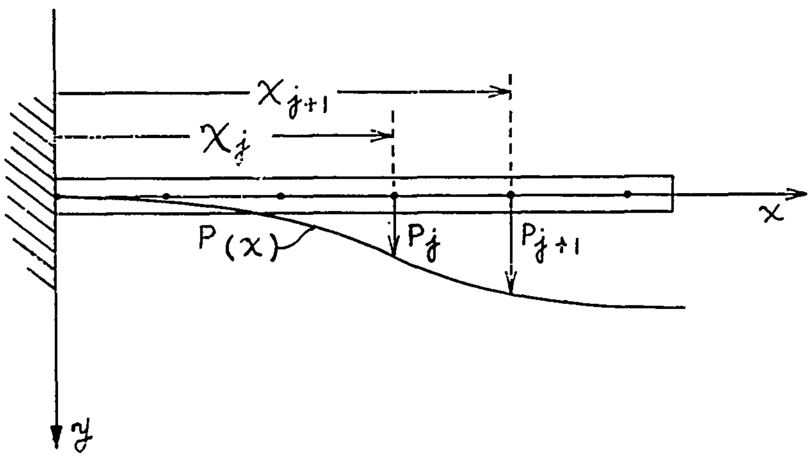
$$
+
\begin{bmatrix} [K_{x'x'}] & [K_{x'y'}] & [K_{x'z'}] \\ [K_{y'x'}] & [K_{y'y'}] & [K_{z'z'}] \\ [K_{z'x'}] & [K_{z'y'}] & [K_{z'z'}] \end{bmatrix}
\begin{Bmatrix} \{P_{x'}\} \\ \{P_{y'}\} \\ \{P_{z'}\} \end{Bmatrix}
=
\begin{Bmatrix} \{Q_{x'}\} \\ \{Q_{y'}\} \\ \{Q_{z'}\} \end{Bmatrix}
+ \sum_j \sigma_j
\begin{bmatrix} \left\{\frac{\partial F_j}{\partial P_{x'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{y'}}\right\} \\ \left\{\frac{\partial F_j}{\partial P_{z'}}\right\} \end{bmatrix}_i
$$

$$
i = 1, 2, \cdots, N
$$

APPENDIX D

INTERPOLATION SCHEMES

The method of analysis in this report requires that the elastic displacements of a continuous system be approximated by displacements at a finite number of discrete points. This numerical approach is general in that it may be applied to bodies of any shape. In order that the forms remain simple, the illustration here will be confined to one-dimensional motion of a thin beam. The purpose of the scheme is to make use of an interpolation formula to relate the continuous displacement at all points to the discrete displacement at neighboring points.



The region between panel points j and $j+1$ is referred to as the j -th bay. Local bay coordinates are defined by

$$\xi = \frac{x - x_j}{x_{j+1} - x_j} \qquad x_j \leq x \leq x_{j+1} \qquad \text{(D-1)}$$

Hence ξ varies between zero and one.

Various interpolation schemes are as follows.

Lumped Mass

This method is not properly an interpolation scheme but is included since it is the most easily used method. A row of panel points is laid along the center-line of the beam. In the region between each pair of panel points, the mass and

center of mass are calculated. This mass is then beamed out to the pair of panel points. When this process is completed, the mass of the beam is represented by a series of lumped masses at the panel points. For this case, the mass matrix is diagonal; the elements along the diagonal are the lumped masses.

Trapezoidal Rule

This rule is called a two-point rule because the displacements in a bay are defined in terms of the displacements at the two neighboring panel points:

$$P(x) = P_j + \frac{x - x_j}{x_{j+1} - x_j}(P_{j+1} - P_j) \tag{D-2}$$

In terms of local bay coordinates, this may be written

$$P(\xi) = (1-\xi)P_j + \xi P_{j+1} \tag{D-3}$$

This rule is useful in calculating mass matrices which do not have to be highly accurate but are wished to be more representative of the system than would be a lumped mass approximation. This rule is also somewhat limited; it cannot be used to calculate the beam curvature as there is no second derivative of P .

The kinetic energy of a beam may be written

$$T = \frac{1}{2}\int_0^L \mathring{P}(x)^2 \rho(x)\,dx \tag{D-4}$$

where $\rho(x)$ is the mass per unit length along the beam. This may be written in the form

$$T = \frac{1}{2}\sum_j \ell_j \int_0^1 \mathring{P}(\xi)^2 \rho(\xi)\,d\xi \tag{D-5}$$

where ℓ_j is the length of the j -th bay between panel points j , $j+1$. The displacement may be written in the form

$$P(\xi) = \begin{Bmatrix} 1 \\ \xi \end{Bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} P_j \\ P_{j+1} \end{Bmatrix} \tag{D-6}$$

which yields

$$\dot{P}^2(\xi) = \begin{Bmatrix} \dot{P}_j \\ \dot{P}_{j+1} \end{Bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \xi \\ \xi & \xi^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{P}_j \\ \dot{P}_{j+1} \end{Bmatrix} \tag{D-7}$$

The kinetic energy in the j -th bay is then

$$T_j = \frac{1}{2} \begin{Bmatrix} \dot{P}_j \\ \dot{P}_{j+1} \end{Bmatrix}' [\bar{a}_j] \begin{Bmatrix} \dot{P}_j \\ \dot{P}_{j+1} \end{Bmatrix} \tag{D-8}$$

where

$$[\bar{a}_j] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \left(\ell_j \int_0^1 \begin{bmatrix} 1 & \xi \\ \xi & \xi^2 \end{bmatrix} \rho(\xi)\, d\xi \right) \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \tag{D-9}$$

The displacements P_j, P_{j+1} may be picked out of all the displacements by an operation of the form

$$\begin{Bmatrix} P_j \\ P_{j+1} \end{Bmatrix} = [B_j]\{P\} \tag{D-10}$$

where the matrix is composed of zeroes except for unity in two elements. The kinetic energy is then

$$T = \frac{1}{2} \{\dot{P}\}' \left(\sum_j [B_j]' [\bar{a}_j][B_j] \right) \{\dot{P}\} \tag{D-11}$$

or

$$T = \frac{1}{2} \{\dot{P}\}' [A] \{\dot{P}\} \tag{D-12}$$

The mass matrix obtained by this process will have non-zero elements on the diagonal and in the first positions off the diagonal.

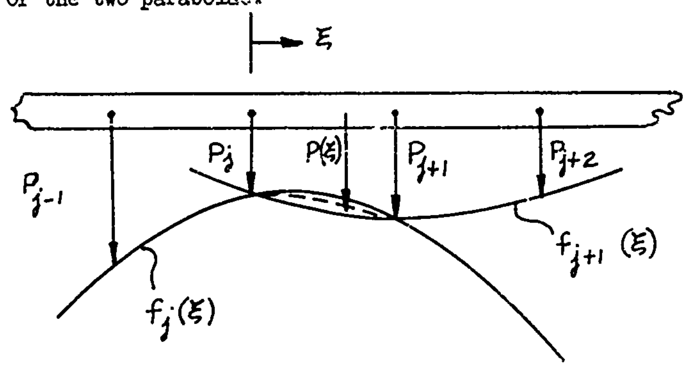
Many interpolation schemes have been devised which make use of an increased number of neighboring panel point displacements. Cubic rules make use of three of these, and four-point rules use two on each side of the local bay. Of the four-point rules in existence, one will be chosen for description. This is referred to as the diparabolic formula. It is felt to be superior to the others since continuity of slope is preserved between bays.

Diparabolic Scheme

If $f_j(\xi)$ is the parabola drawn through the set of points at x_{j-1}, x_j, and x_{j+1} and $f_{j+1}(\xi)$ is the parabola drawn through the set of points at x_j, x_{j+1}, and x_{j+2}, then

$$P(\xi) = (1-\xi) f_j(\xi) + \xi f_{j+1}(\xi) \qquad 0 \leq \xi \leq 1 \tag{D-13}$$

is the formula for interpolation in the j-th bay. Thus the formula is a weighted average of the two parabolas.



This may be described in terms of the displacements at the panel points. The parabolas

$$f_j(\xi) = a_j^{(1)} + a_j^{(2)}\xi + a_j^{(3)}\xi^2 \tag{D-14}$$

$$f_{j+1}(\xi) = a_{j+1}^{(1)} + a_{j+1}^{(2)}\xi + a_{j+1}^{(3)}\xi^2 \tag{D-15}$$

are defined by

$$f_j(\xi_{j-1}) = P_{j-1}, \quad f_j(\xi_j) = P_j, \quad f_j(\xi_{j+1}) = P_{j+1} \tag{D-16}$$

$$f_{j+1}(\xi_j) = P_j, \quad f_{j+1}(\xi_{j+1}) = P_{j+1}, \quad f_{j+1}(\xi_{j+2}) = P_{j+2} \tag{D-17}$$

The constants, a_j, may be evaluated in terms of the panel point deflections by combining the above forms. They may be written

$$\begin{Bmatrix} P_{j-1} \\ P_j \\ P_{j+1} \end{Bmatrix} = \begin{bmatrix} 1 & \xi_{j-1} & \xi_{j-1}^2 \\ 1 & \xi_j & \xi_j^2 \\ 1 & \xi_{j+1} & \xi_{j+1}^2 \end{bmatrix} \begin{Bmatrix} a_j^{(1)} \\ a_j^{(2)} \\ a_j^{(3)} \end{Bmatrix} \tag{D-18}$$

and

$$\begin{Bmatrix} P_j \\ P_{j+1} \\ P_{j+2} \end{Bmatrix} = \begin{bmatrix} 1 & \xi_j & \xi_j^2 \\ 1 & \xi_{j+1} & \xi_{j+1}^2 \\ 1 & \xi_{j+2} & \xi_{j+2}^2 \end{bmatrix} \begin{Bmatrix} a_{j+1}^{(1)} \\ a_{j+1}^{(2)} \\ a_{j+1}^{(3)} \end{Bmatrix} \tag{D-19}$$

The solution of these equations is written in terms of the inverse matrices

$$[\Lambda]_j = \begin{bmatrix} 1 & \xi_{j-1} & \xi_{j-1}^2 \\ 1 & \xi_j & \xi_j^2 \\ 1 & \xi_{j+1} & \xi_{j+1}^2 \end{bmatrix}^{-1} \tag{D-20}$$

$$[\Lambda]_{j+1} = \begin{bmatrix} 1 & \xi_j & \xi_j^2 \\ 1 & \xi_{j+1} & \xi_{j+1}^2 \\ 1 & \xi_{j+2} & \xi_{j+2}^2 \end{bmatrix}^{-1} \tag{D-21}$$

so that

$$\begin{Bmatrix} a_j^{(1)} \\ a_j^{(2)} \\ a_j^{(3)} \end{Bmatrix} = [\Lambda]_j \begin{Bmatrix} P_{j-1} \\ P_j \\ P_{j+1} \end{Bmatrix} \tag{D-22}$$

$$\begin{Bmatrix} a_{j+1}^{(1)} \\ a_{j+1}^{(2)} \\ a_{j+1}^{(3)} \end{Bmatrix} = [\Lambda]_{j+1} \begin{Bmatrix} P_j \\ P_{j+1} \\ P_{j+2} \end{Bmatrix} \tag{D-23}$$

Formulas for the parabolas are then

$$f_j(\xi) = \{1\ \ \xi\ \ \xi^2\}\begin{Bmatrix} a_j^{(1)} \\ a_j^{(2)} \\ a_j^{(3)} \end{Bmatrix} = \{1\ \ \xi\ \ \xi^2\}[\Lambda]_j \begin{Bmatrix} P_{j-1} \\ P_j \\ P_{j+1} \end{Bmatrix} \tag{D-24}$$

$$f_{j+1}(\xi) = \{1\ \ \xi\ \ \xi^2\}\begin{Bmatrix} a_{j+1}^{(1)} \\ a_{j+1}^{(2)} \\ a_{j+1}^{(3)} \end{Bmatrix} = \{1\ \ \xi\ \ \xi^2\}[\Lambda]_{j+1} \begin{Bmatrix} P_j \\ P_{j+1} \\ P_{j+2} \end{Bmatrix} \tag{D-25}$$

The diparabolic formula is then

$$P(\xi) = \{1\ \ \xi\ \ \xi^2\}\left((1-\xi)[\Lambda]_j \begin{Bmatrix} P_{j-1} \\ P_j \\ P_{j+1} \end{Bmatrix} + \xi[\Lambda]_{j+1} \begin{Bmatrix} P_j \\ P_{j+1} \\ P_{j+2} \end{Bmatrix} \right) \tag{D-26}$$

which is of the form

$$P(\xi) = \{1\ \ \xi\ \ \xi^2\ \ \xi^3\}[\delta]_j \begin{Bmatrix} P_{j-1} \\ P_j \\ P_{j+1} \\ P_{j+2} \end{Bmatrix} \qquad \begin{matrix} 0 \le \xi \le 1 \\ x_j \le x \le x_{j+1} \end{matrix} \tag{D-27}$$

where

$$[\delta]_j = \begin{bmatrix} [\Lambda]_j & \\ & 0 \end{bmatrix} - \begin{bmatrix} & 0 \\ [\Lambda]_j & \end{bmatrix} + \begin{bmatrix} 0 & \\ & [\Lambda]_{j+1} \end{bmatrix} \tag{D-28}$$

Numerically, for equal intervals,

$$[\Lambda]_j = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \tag{D-29}$$

$$[\Lambda]_{j+1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \tag{D-30}$$

$$[\delta] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1 & 1/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3/2 & 2 & -1/2 \\ 0 & 1/2 & 1 & 1/2 \end{bmatrix}$$

$$[\delta] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1 & -5/2 & 2 & -1/2 \\ -1/2 & 3/2 & -3/2 & 1/2 \end{bmatrix} \tag{D-31}$$

The kinetic energy of a beam is written

$$T = \frac{1}{2}\int_0^L \dot{P}(x)^2 \rho(x)\,dx \tag{D-32}$$

where $\rho(x)$ is the mass per unit length.

In the j -th bay,

$$\dot{P}(\xi)=\{1 \;\; \xi \;\; \xi^2 \;\; \xi^3\}[S]_j\begin{Bmatrix}\dot{P}_{j-1}\\ \dot{P}_j\\ \dot{P}_{j+1}\\ \dot{P}_{j+2}\end{Bmatrix} \tag{D-33}$$

$$\dot{P}(\xi)^2=\begin{Bmatrix}\dot{P}_{j-1}\\ \dot{P}_j\\ \dot{P}_{j+1}\\ \dot{P}_{j+2}\end{Bmatrix}'[S]_j'\begin{bmatrix}1 & \xi & \xi^2 & \xi^3\\ \xi & \xi^2 & \xi^3 & \xi^4\\ \xi^2 & \xi^3 & \xi^4 & \xi^5\\ \xi^3 & \xi^4 & \xi^5 & \xi^6\end{bmatrix}[S]_j\begin{Bmatrix}\dot{P}_{j-1}\\ \dot{P}_j\\ \dot{P}_{j+1}\\ \dot{P}_{j+2}\end{Bmatrix} \tag{D-34}$$

The kinetic energy of the j -th bay is then

$$T_j=\frac{1}{2}\begin{Bmatrix}\dot{P}_{j-1}\\ \dot{P}_j\\ \dot{P}_{j+1}\\ \dot{P}_{j+2}\end{Bmatrix}'[\bar{a}]_j\begin{Bmatrix}\dot{P}_{j-1}\\ \dot{P}_j\\ \dot{P}_{j+1}\\ \dot{P}_{j+2}\end{Bmatrix} \tag{D-35}$$

where

$$[\bar{a}]_j=[S]_j'\,\ell_j\int_0^1\begin{bmatrix}1 & \xi & \xi^2 & \xi^3\\ \xi & \xi^2 & \xi^3 & \xi^4\\ \xi^2 & \xi^3 & \xi^4 & \xi^5\\ \xi^3 & \xi^4 & \xi^5 & \xi^6\end{bmatrix}\rho(\xi)\,d\xi\,[S]_j \tag{D-36}$$

If the displacements in the j -th bay are picked from all the displacements by an operation of the form

$$\begin{Bmatrix} P_{j-1} \\ P_j \\ P_{j+1} \\ P_{j+2} \end{Bmatrix} = [B]_j \{P\} \tag{D-37}$$

where $[B]_j$ is a matrix whose elements are one or zero, then one may write

$$T = \frac{1}{2}\{\dot{P}\}'\left(\sum_j [B]_j' [\bar{a}]_j [B]_j\right)\{\dot{P}\} \tag{D-38}$$

or

$$T = \frac{1}{2}\{\dot{P}\}'[A]\{\dot{P}\} \tag{D-39}$$

BIBLIOGRAPHY

1. Goldstein, Herbert, Classical Mechanics, Addison - Wesley Press, Inc. 1950.

2. Seely, F. B. and Smith, J. O., Advanced Mechanics of Materials, John Wiley and Sons, Inc. 1957.

3. Levy, S., "Influence Coefficients of Tapered Cantilever Beams Computed on SEAC", Journal of Applied Mechanics, March 1953, Vol. 20, No. 1

4. Bisplinghoff, R. L., Ashley, and Halfman, R. L., Aeroelasticity, Addison-Wesley Press, Inc. 1955.

5. Abramson, H. Norman, The Dynamics of Airplanes, The Ronald Press Co., 1958

6. USAF Parachute Handbook, WADC TR-55-265, December 1956

7. Nelson, F. R., Koerner, W., Trudel, R. E., Dynamics of the Airframe, Vol. 2, BuAer Report AR-61-4, Northrop Aircraft, Inc. 1953.

8. Hadekel, R., "The Mechanical Characteristics of Pneumatic Tyres", S & T Memo. No. 5/50. 1950.

9. Hample, W. G., "Frictional Study of Aircraft Tire Material on Concrete", NACA TN 3294, 1955.

10. Luthman, R. R., "Coefficient of Friction of Aircraft Tires on Concrete Runways", WADC TR 55-179, 1955.

11. Gough, Sawyer, and Trant, "Tire-Runway Braking Friction Coefficients", NATO Advisory Group for Aeronautical R & D Report 51, 1956.

12. Flügge, W. and Coale, C. W., "The Influence of Wheel Spinup on Landing Gear Impact", NACA TN 3217, 1954.

13. Thorson, K. R., "A Rational Method for Predicting Tire Cornering Forces and Lateral Stiffness", Boeing Airplane Co. D-11719, 1951.

14. Bernhard, R. K., and Finelli, J., "Pilot Studies on Soil Dynamics", ASTM Special Technical Publication No. 156, 1954.

15. Esgar, J. B., and Morgan. W. C., "Analytical Study of Soft Landings on Gas-filled Bags." NASA TR R-75, 1960.

16. Walls, J. H., "An Experimental Study of Orifice Coefficients, Internal Strut Pressures, and Loads on a Small Oleo-Pneumatic Shock Strut", NACA TN3426, 1955.

17. Walls, J. H., "Investigation of the Air-Compression Process during Drop Tests of an Oleo-Pneumatic Landing Gear", NACA TN 2477, 1951.

18. Frazer, R. A., Duncan, D. Sc., and Collar, A.M., Elementry Matrices, Cambridge: At the University Press, 1946

19. Harris, C. M. and Crede, C. E., Shock and Vibration Handbook , Vol. 2, McGraw-Hill, 1961